National Height Datum, the Gauss-Listing Geoid Level Value w_0 and Its Time Variation $\dot{w}_0$

(Baltic Sea Level Project: Epoch 1990.8, 1993.8, 1997.4)

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Abstract. A methodology for precise determination of the fundamental *geodetic parameter* w_0, the potential value of *Gauss-Listing* geoid, as well as its *time derivative* $\dot{w}_0$ is presented. The method is based on: *(i)* ellipsoidal harmonic expansion of the external gravitational field of the Earth to degree/order 360/360 (130321 coefficients http://www.uni-stuttgart.de/gi/research/index.html-#Projects) with respect to the *International Reference Ellipsoid* WGD2000, at the GPS positioned stations. *(ii)* ellipsoidal free-air gravity reduction of degree/order 360/360 , based on orthometric heights of the GPS positioned stations. The method has been numerically tested for the data of three GPS campaigns of *Baltic Sea Level Projects* (epoch 1990.8, 1993.4, and 1997.4). New w_0 and $\dot{w}_0$ values ($w_0 = (62636855.75 \pm\ 0.21)(m^2/s^2)$, $\dot{w}_0 = (-0.0099 \pm 0.00079)(m^2/s^2)/year$, $\omega_0/\overline{\gamma} = 6379781.502(m)$, $\dot{w}_0/\overline{\gamma} =\ 1.0(mm/year)$, and $\overline{\gamma} = -9.81802523m/s^2$) for the test region (Baltic Sea) have been obtained. As by-products of the main study we also succeeded *(i)* to determine the high-resolution *Sea Surface Topography* map for the Baltic Sea, *(ii)* to determine the most accurate regional geoid amongst four different regional *Gauss-Listing geoids* currently proposed for the *Baltic Sea*, and *(iii)* to determine the differences between *National Height Datum* of countries around the *Baltic Sea*.

Keywords: Potential value of geoid. Time variations of the potential value of geoid. Sea Surface Topography. Differences between National Height Datums.

0 Introduction

There are four quantities, namely $\{w_0, gm, j_2, \omega\}$, which are currently accepted by geodetic community, as the fundamental geodetic parameters, as documented by E. *Groten* (2000), for instance. Among these parameters, w_0 plays a crucial role in the geoid determination as well as the computation of the best fitting reference equipotential surface to the geoid. For example, the reference potential field $W(r) = gm/r = w_0$ gauged to the geoid potential value w_0, leads to the sphere $\mathcal{S}^2_{R=gm/w_0}$ as the best fitting reference equipotential surface to *geoid*, or the *Somigliana-Pizzetti potential field* gauged to the geoid potential value w_0 generates the ellipsoid of revolution $\mathcal{E}^2_{a,a,b}$, i.e. WGD2000 (*E. Grafarend and A. Ardalan* 1999), as the best fitting reference equipotential surface to the geoid. According to *E. Grafarend and A. Ardalan* (1999, page 614), the accuracy of such a best-fitting ellipsoid is mainly driven by the accuracy of w_0.

Here we are proposing the following algorithm for the precise determination of fundamental geodetic parameter w_0:

(i) determine the gravity potential at the accurately positioned GPS stations, by using the ellipsoidal harmonic expansion of degree/order 360/360 (130321 coefficients http://www.uni-stuttgart.de/gi/research/index.html#Projects), plus the ellipsoidal centrifugal potential.

(ii) reduce the derived potential values to the *Mean Sea Level* (MSL) via the precise ortho-metric height of GPS positioned stations, and the ellipsoidal free air gravity reduction of degree/order 360/360.

The input data for the "w_0 operational procedure" quoted above can be obtained by GPS observations at permanent tide gauges or at stations which are located *close to* tide gauge stations and are well connected to the tide gauge stations by precise levelling. The GPS observations collected for *Baltic Sea Level Project* (see for instance, *J. Kakkuri,* (1990, 1995) and *M. Poutanen and J. Kakkuri,* (1999)) provide us with such a source of information. *Figure 0–1* presents a typical configuration of

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the GPS and permanent tide gauge stations of *Baltic Sea Level Project*.

Intuitively speaking, the global geopotential model together with position information derived by GPS provide us with the global gravity information, and, in contrast, the connection of GPS stations to the tide gauge by precise levelling, supplies us with the *Mean Sea Level* information. Recalling that the geoid according to the Gauss / *Listing* definition (*C. Gauss* 1828 page 49, *J. Listing* 1873 page 45) is an equipotential surface which fits to the *Mean Sea Level* in an optimal way, the approach outlined above should be adequate for a global evaluation of w_0. However, since the tide gauge observations, aside from random errors, are always affected by some local effects, we may obtain different w_0 values for different tide gauge stations. However, the average value of w_{0i} data (the index i runs from one to the total number of the tide gauge stations) computed from a set of globally well distributed tide gauges may be considered an unbiased or, at least, a minimum biased estimate of the geoid potential value w_0

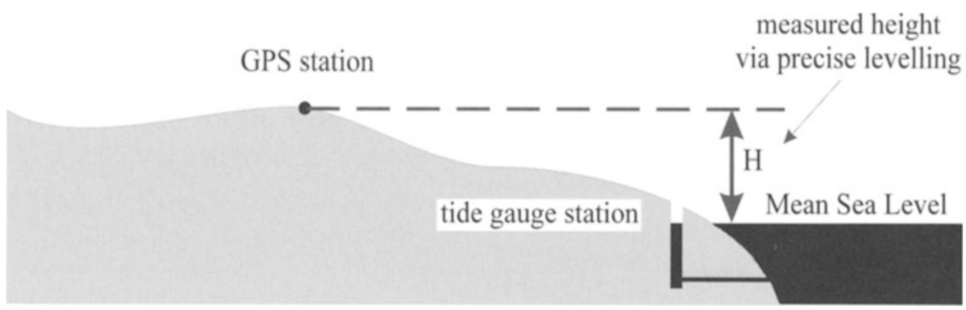

Figure 0–1: The GPS station, permanent tide gauge station, and measured precise orthometric height of the GPS station above Mean Sea Level.

In *Section 1*, we set up the theoretical background of our w_0 computation methodology. In *Section* 2 we shall introduce those data that we will use for various numerical computations. *Sections 3, 4* and *5* focus on the results of w_0 computation based on GPS data of *Baltic Sea Level Project*, 1st campaign, 2nd campaign, and 3rd campaign. *Section 5*, leads to the best estimate of $w_0 = (62636855.75 \pm 0.21)(m^2/s^2)$ with a remarkably small mean square error. In *Section 6*, we aim at an estimate of the time derivative of w_0, namely $\dot{w}_0 = (0.0099 \pm 0.00079)\ (m^2/s^2)$/year. *Section 7* discusses the apparent difference in the height datum between various countries around the *Baltic Sea* as well as the transfer function of gravity potential differences into geometric heights in order to obtain the differences between height datums. For the *unification* of national, regional or continental to a *global height datum* we acknowledge contributions of *R. Rummel and P. Teunissen* (1988), *B. Heck, and R. Rummel* (1990), *P. Xu and R. Rummel* (1991), *P. Xu* (1992), *R. Rapp* (1994), *R. Rummel and K. Ilk* (1995), and *F. Sansò and S. Usai* (1995). Finally *Section 8* concentrates on the determination of *Sea Surface Topography* of Baltic Sea at the tide gauges of the *Baltic Sea Level Project*.

1. W_0 / W_{0i} Computations: Problem Formulation

It is well known that a *slowly* uniformly rotating, self-gravitating liquid body of radial mass distribution, forms in its equilibrium state an *ellipsoid of revolution* at its boundary. Therefore, the Earth as a massive body, which once was a liquid at hydrostatic equilibrium, globally does not deviate considerably from an *oblate spheroid* or *ellipsoid of revolution*. Therefore, the external gravity field of the Earth can most conveniently be presented in terms of an ellipsoidal model as explained in *Table 1–1*.

Table 1–1: Additive decomposition of the gravity potential W into gravitational potential U and the centrifugal potential V.

$$W(\lambda,\phi,u) = U(\lambda,\phi,u) + V(\lambda,\phi,u) \tag{1.1}$$

where

$$U(\lambda,\phi,u) = \sum_{n=0}^{\infty}\sum_{m=-n}^{+n} \frac{Q^*_{n|m|}(\frac{u}{\varepsilon})}{Q^*_{n|m|}(\frac{b}{\varepsilon})} e_{nm}(\lambda,\phi) u_{nm}\,, \tag{1.2}$$

$$\begin{aligned} V(\phi,u) &= \frac{1}{2}\omega^2(x^2+y^2) = \frac{1}{2}\omega^2(u^2+\varepsilon^2)\cos^2\phi \\ &= \frac{1}{3}\omega^2(u^2+\varepsilon^2)(e_{00} - \frac{1}{\sqrt{5}} e_{20}(\lambda,\phi)) \end{aligned}\,, \tag{1.3}$$

and

$$e_{nm}(\lambda,\phi) = P^*_{n|m|}(\sin\phi)\begin{bmatrix} \cos m\lambda & \forall m \geq 0 \\ \sin \mid m \mid \lambda & \forall m < 0 \end{bmatrix}. \tag{1.4}$$

According to *Table 1–1*, we have additively decomposed the gravity potential W into the gravitational potential U and the centrifugal potential V; both represented in terms of *Jacobi ellipsoidal coordinates* $\{\lambda,\phi,u\}$ also called *spheroidal.* In the space external to the Earth, its gravitational potential is *harmonic*, namely it is an element of the three-dimensional *Laplace equation* with *associated Legendre functions* of the first $P^*_{nm}(\sin\phi)$ and of the *second kind* $Q^*_{nm}(u/b)$ as well as $\{\cos m\lambda$, $\sin \mid m \mid \lambda\}$ as *eigenfunctions*. Accordingly, in terms of surface spheroidal harmonics $e_{nm}(\lambda,\phi)$ given by (1.4), the equations (1.1)–(1.3) are proper representations. For the definition of the *Jacobi ellipsoidal coordinates* $\{\lambda,\phi,u\}$ we refer to *Appendix A*. *Appendix B* provides us with the definition of the normalised associated Legendre functions of the first as well as of the second kind. Finally, *Appendix C* outlines how the *ellipsoidal harmonic coefficients* u_{nm} can be obtained from the available spherical harmonic coefficients.

Equation (1.1) is exact and without any approximation. However, in practice we have to approximate it by using a limited number of ellipsoidal harmonic coefficients, say up to degree/order 360/360. Consequently, we arrive at an approximate form of (1.1), which can serve as a reference gravity field of the Earth. Such a reference field covers the gravity effects of the features which are bigger than 50–60km.

Roughly speaking, now if we place the GPS receiver at point p_0 on the Mean Sea Level (i.e. geoid) and measure the Cartesian coordinates $\{x_0,y_0,z_0\}$ of p_0, then once we have converted these Cartesian coordinates into Jacobi ellipsoidal coordinates $\{\lambda_0,\phi_0,u_0\}$ (via the transformation relations given in *Appendix A*) an estimation of geoid potential value w_0 can be obtained according to *Table 1–2*.

Table 1–2: High degree/order reference potential field of the external gravity field of the Earth.

$$\begin{aligned} w_0 \doteq W(\lambda_0,\phi_0,u_0) &= \sum_{n=0}^{360}\sum_{m=-n}^{+n} \frac{Q^*_{n|m|}(\frac{u_0}{\varepsilon})}{Q^*_{n|m|}(\frac{b}{\varepsilon})} e_{nm}(\lambda_0,\phi_0) u_{nm} \\ &+\frac{1}{3}\omega^2(u_0^2+\varepsilon^2)(e_{00} - \frac{1}{\sqrt{5}} e_{20}(\lambda_0,\phi_0)) \end{aligned} \tag{1.5}$$

Since at the geoid's level the geometrical variation of the equipotential is quite smooth, degree/order 360/360 is enough for the computation of the geoid potential value at the level of accuracy better than $0.5 m^2/s^2$. Note that the high-frequency changes of the gravity field is mainly due to the topographical masses, therefore at the geoid's level we do not have any high-frequency variations

and as such the ellipsoidal harmonic expansion of degree/order 360/360 can satisfactorily represent the geoid potential value w_0.

In practice however, we may place the GPS receive at point p, which is at the *vicinity* of a tide gauge station, and measure the Cartesian coordinates $\{x, y, z\}$. These Cartesian coordinates once converted into Jacobi ellipsoidal coordinates $\{\lambda, \phi, u\}$ can provide us with the gravity potential $W(\lambda, \phi, u)$ at the point p, which is of course different from the geoid potential value w_0.

Knowing, that the potential difference between the points $p\{\lambda, \phi, u\}$ and the point $p_0\{\lambda_0, \phi_0, u_0\}$ on the geoid is only due to the orthometric height of $p\{\lambda, \phi, u\}$, the geoid potential value w_0 can be written in terms of the Taylor series expansion around the potential value $W(\lambda, \phi, u)$ at the point $p\{\lambda, \phi, u\}$ as explained in *Table 1–3*.

Table 1–3: Taylor series expansion of the geoid potential value w_0 around the potential value $W(\lambda, \phi, u)$ at the point $p\{\lambda, \phi, u\}$.

"Taylor series expansion of the geoid potential value w_0"

$$\begin{aligned} w_0 &= W(\lambda, \phi, u) + \frac{1}{1!}\mathrm{D}_u W(\lambda, \phi, u) \cdot (u_0 - u) \\ &+ \frac{1}{2!}\mathrm{D}_u(\mathrm{D}_u W(\lambda, \phi, u))(u_0 - u)^2 \\ &+ \mathcal{O}((u_0 - u)^3) \end{aligned} \tag{1.6}$$

"Partial derivative of potential value W along the coordinate line of u"

$$\mathrm{D}_u W = \frac{\partial}{\partial u} W = \sqrt{g_{uu}} \nabla_{\mathbf{e}_u} W \tag{1.7}$$

"Directional derivatPive of potential value W along the coordinate line of u"

$$\begin{aligned} \nabla_{\mathbf{e}_u} W &:= \langle \operatorname{grad} W(\lambda, \phi, u) \mid \mathbf{e}_u \rangle \\ &= \left\langle \mathbf{e}_\lambda \frac{1}{\sqrt{g_{\lambda\lambda}}} \frac{\partial W}{\partial \lambda} + \mathbf{e}_\phi \frac{1}{\sqrt{g_{\phi\phi}}} \frac{\partial W}{\partial \phi} + \mathbf{e}_u \frac{1}{\sqrt{g_{uu}}} \frac{\partial W}{\partial u} \,\middle|\, \mathbf{e}_u \right\rangle \\ &= \frac{1}{\sqrt{g_{uu}}} \frac{\partial W(\lambda, \phi, u)}{\partial u} \end{aligned} \tag{1.8}$$

"Taylor series expansion of the geoid potential value w_0 in terms of directional derivative operator"

$$\begin{aligned} w_0 &= W(\lambda, \phi, u) + \frac{1}{1!} \nabla_{\mathbf{e}_u} W(\lambda, \phi, u) \cdot \sqrt{g_{uu}}(u_0 - u) \\ &= W(\lambda, \phi, u) + \nabla_{\mathbf{e}_u}(U(\lambda, \phi, \eta) + V(\lambda, \phi, \eta)) \cdot \sqrt{g_{uu}}(u_0 - u) \\ &= W(\lambda, \phi, u) + \frac{1}{\sqrt{g_{uu}}} \frac{\partial(U(\lambda, \phi, u) + V(\lambda, \phi, u))}{\partial u} \cdot \sqrt{g_{uu}}(u_0 - u) \\ &= W(\lambda, \phi, u) + \frac{1}{\sqrt{g_{uu}}} \left(\frac{\partial U(\lambda, \phi, u)}{\partial u} + \frac{\partial V(\lambda, \phi, u)}{\partial u} \right) \cdot \Delta u^{(1)} \\ &= W(\lambda, \phi, u) + \Delta W(\lambda, \phi, u) \end{aligned} \tag{1.9}$$

subject to

$$\frac{\partial U}{\partial u} = \sum_{n=0}^{360} \sum_{m=-n}^{+n} u_{nm} \frac{1}{Q^*_{n|m|}(\frac{b}{\varepsilon})} \frac{\partial Q^*_{n|m|}(\frac{u}{\varepsilon})}{\partial u} e_{nm}(\lambda, \phi), \tag{1.10}$$

$$\frac{\partial V}{\partial u} = \omega^2 u \sqrt{u^2 + \varepsilon^2} \cos^2 \phi, \tag{1.11}$$

$$\sqrt{g_{uu}} = \frac{\sqrt{u^2 + \varepsilon^2 \sin^2 \phi}}{\sqrt{u^2 + \varepsilon^2}}. \tag{1.12}$$

In *Table 1–3*, $\mathrm{D}_u W$ is the partial derivative of potential $W(\lambda,\phi,u)$ with respect to u, and $(u_0 - u)$ is the height of GPS station above MSL, of course in terms of Jacobi ellipsoidal height component. The partial derivative $\mathrm{D}_u W$ is related to directional derivative of W along the coordinate line of u as introduced by (1.7). For the definition of directional derivative $\nabla_{\mathbf{e}_u} W$ along the coordinate line of u, we refer to equation (1.8). $\mathbf{e}_\lambda = \mathbf{t}_\lambda / \sqrt{g_{\lambda\lambda}}$, $\mathbf{e}_\phi = \mathbf{t}_\phi / \sqrt{g_{\phi\phi}}$, and $\mathbf{e}_u = \mathbf{t}_u / \sqrt{g_{uu}}$ are orthonormal base vectors of the Jacobi ellipsoidal coordinates $\{\lambda,\phi,u\}$, and $\sqrt{g_{\lambda\lambda}}$, $\sqrt{g_{\phi\phi}}$, and $\sqrt{g_{uu}}$ are metric tensors. See *Appendix A, equation (A.10)* for the definition of the metric tensors in terms of Jacobi ellipsoidal coordinates $\{\lambda,\phi,u\}$. The vectors $\mathbf{t}_\lambda = \partial\mathbf{x}/\partial\lambda$, $\mathbf{t}_\phi = \partial\mathbf{x}/\partial\phi$, $\mathbf{t}_u = \partial\mathbf{x}/\partial u$ are tangent to coordinate lines of Jacobi ellipsoidal coordinates λ, ϕ, and u, respectively. Therefore, (1.6) up to the terms of the order of magnitude $\mathcal{O}((u_0 - u)^2)$ can be written as (1.9).

According to *C. Eringen* (1962, page 437) $\Delta u^{(1)} = \sqrt{g_{uu}}(u_0 - u)$ is the physical component of the Jacobi ellipsoidal height difference $(u_0 - u)$. Partial derivative of associated Legendre functions of second kind with respect to u can be derived from the recursive relations of *Table 1–4*.

Table 1–4: Partial derivative of associated Legendre functions of second kind with respect to u.

"Recursive relations"

$$
\begin{aligned}
\frac{\partial Q^*_{0,0}(\frac{u}{\varepsilon})}{\partial u} &= -\frac{a}{\sqrt{u^2+\varepsilon^2}} \\
\frac{\partial Q^*_{n,0}(\frac{u}{\varepsilon})}{\partial u} &= n\frac{u}{\sqrt{u^2+\varepsilon^2}}Q^*_{n,0}(\frac{u}{\varepsilon}) - (2n+1)\frac{a}{\sqrt{u^2+\varepsilon^2}}Q^*_{n-1,0}(\frac{u}{\varepsilon}) \\
\frac{\partial Q^*_{n,m}(\frac{u}{\varepsilon})}{\partial u} &= -(n-m+1)\,Q^*_{n,m-1}(\frac{u}{\varepsilon}) - m\frac{u}{\sqrt{u^2+\varepsilon^2}}Q^*_{n,m}(\frac{u}{\varepsilon}) \;\; \forall m\in[1,n]
\end{aligned}
\tag{1.13}
$$

We call the second term of (1.9), i.e. $\Delta W(\lambda,\phi,u)$, the "*ellipsoidal free-air reduction*" of the potential value $W(\lambda,\phi,u)$ to the geoid's surface. Since in the ellipsoidal harmonic expansion all masses of the Earth are condensed inside the reference ellipsoid of WGD2000, in the eyes of ellipsoidal harmonic model, we have actually *free-air* between p and p_0! The physical component $\sqrt{g_{uu}}(u_0 - u)$ here can be interpreted as the orthometric height H_0 of the GPS station with opposite sign, i.e. $\sqrt{g_{uu}}(u_0 - u) = -H_0$.

Therefore, the geoid potential value w_0 can be derived from the Jacobi coordinates $\{\lambda,\phi,u\}$ of the GPS station p and the orthometric height of p, H_0^p, as outlined in *Table 1–5.*

Table 1–5: Operational procedure for computing the geoid potential value w_0

$$w_0 = \sum_{n=0}^{360}\sum_{m=-n}^{+n} u_{nm} \frac{Q^*_{n|m|}(\frac{u}{\varepsilon})}{Q^*_{n|m|}(\frac{b}{\varepsilon})} e_{nm}(\lambda,\phi) + \frac{1}{2}\omega^2(u^2+\varepsilon^2)\cos^2\phi$$
$$+\frac{1}{\sqrt{g_{uu}}}\Big(\sum_{n=0}^{360}\sum_{m=-n}^{+n} u_{nm} \frac{1}{Q^*_{n|m|}(\frac{b}{\varepsilon})} \frac{\partial Q^*_{n|m|}(\frac{u}{\varepsilon})}{\partial u} e_{nm}(\lambda,\phi) \tag{1.14}$$
$$+\omega^2 u\sqrt{u^2+\varepsilon^2}\cos^2\phi\Big)(-H_0^p)$$
$$= W(\lambda,\phi,u) + \Delta W(\lambda,\phi,u;H_0^p)$$

subject to:

(i) the gravity potential at point p

$$W(\lambda,\phi,u) = \sum_{n=0}^{360}\sum_{m=-n}^{+n} u_{nm} \frac{Q^*_{n|m|}(\frac{u}{\varepsilon})}{Q^*_{n|m|}(\frac{b}{\varepsilon})} e_{nm}(\lambda,\phi) + \frac{1}{2}\omega^2(u^2+\varepsilon^2)\cos^2\phi\,, \tag{1.15}$$

(ii) ellipsoidal free-air reduction of degree/order 360/360

$$\Delta W(\lambda,\phi,u;H_0^p) = \frac{1}{\sqrt{g_{uu}}}\Big(\sum_{n=0}^{360}\sum_{m=-n}^{+n} u_{nm} \frac{1}{Q^*_{n|m|}(\frac{b}{\varepsilon})} \frac{\partial Q^*_{n|m|}(\frac{u}{\varepsilon})}{\partial u} e_{nm}(\lambda,\phi)\,, \tag{1.16}$$
$$+\omega^2 u\sqrt{u^2+\varepsilon^2}\cos^2\phi\Big)(-H_0^p)$$

Having set up the theoretical foundation of the w_0 computation problem, we can begin our case study by the computing w_0 and $\dot{w}_0$ from the GPS observation of the Baltic Sea Level projects, first, second, and third campaigns in the next sections.

2. W$_0$ Computations: Input Data

Based on the method described in previous section here we compute the w_0 value via the GPS observations of three successive GPS campaigns of Baltic Sea Level Project. For a review of the state of the art of Baltic Sea Level project *J. Kakkuri* (1990, 1995), and *M. Poutanen and J Kakkuri* (1999) are recommended. *Table 2–1–Table 2–3* present the Cartesian coordinates of the GPS stations of Baltic Sea level project, 1st, 2nd, and 3rd campaigns, respectively. *Table 2–4–Table 2–6* are presenting the computed Jacobi ellipsoidal coordinates (see *Appendix A* for transformation relations) of those stations given in *Table 2–1–Table 2–3*. The ellipsoidal harmonic coefficients needed for the series expansion of (1.15) and (1.16) are provided via the transformation of the spherical harmonic coefficients of *EGM96* (*F. Lemoine et al.* 1998) to ellipsoidal ones with respect to the reference ellipsoid of *WGD2000* through exact transformation relations given in *Appendix C.* The ellipsoidal harmonic coefficients are computed in *mean-tide permanent-tide system* (see *M. Ekman* (1996) for the definition of various permanent-tide systems). For this purpose we first transferred the second zonal spherical harmonic coefficients of *EGM96* geopotential model from tide free system into mean tide system, and then applied the transformation machinery of *Appendix C* to obtain ellipsoidal harmonic coefficients from spherical harmonic coefficients. We have made available the computed ellipsoidal harmonic coefficients to public, and interested readers can download the ellipsoidal harmonic coefficients plus the manual of using them, from the homepage of Geodetic Institute of the Stuttgart University (http: // www.uni-stuttgart.de / gi / research / index.html # Projects). As the reference ellipsoid we use the *WGD2000* in mean tide system (c.f. *E. Grafarend and A. Ardalan*, 1999). In other words, we performed all the computations in mean-tide permanent-tide system.

We obtained the orthometric heights, of GPS stations from five different sources namely: *M. Poutanen et al.* (1999), *J. Kakkuri* (1995), *J. Kakkuri and M. Poutanen* (1997), *M. Poutanen et al.* (1999), and *J. Kakkuri* (2000). Consequently, we

performed the computations under five cases defined below.

Case 1: the orthometric heights from *M. Poutanen et al.* (1999)
Case 2: the orthometric heights from *J. Kakkuri* (1995)
Case 3: the orthometric heights from *J. Kakkuri and M. Poutanen* (1997)
Case 4: the orthometric heights from *M. Poutanen et al.* (1999)
Case 5: the heights from *Kakkuri* (2000)

In case 1, the orthometric heights H_0 are given in their respective national height systems, measured directly by precise levelling to tide gauges, while for the cases 2–5 orthogonal heights H_0 are derived from GPS ellipsoidal height h and different geoid solutions N proposed for Baltic Sea. Recall that neglecting the curvature of plumb line following relation holds.

$$H_0 \doteq h - N \tag{2.1}$$

It should be mention that the difference between orthometric heights of cases 2–5 is mainly due to different geoid solutions and application of different reductions to obtain mean geoid via polynomial fitting, and using different starting latitudes (see for example *M. Vermmer* (1995) for more details).
In the following sections, the results of w_0 computations for each of the above mentioned cases will be presented.

Table 2–1: Cartesian coordinates of GPS stations of BSL project first campaign 1990.8, in ITRF 91 reference frame.

Station Name	X (m)	Y (m)	Z (m)
Borkum (Ger)	3770668.4100	446076.5560	5107686.4450
Degerby (Fin)	2994005.5910	1112565.2290	5502270.8220
Furuögrund (Swe)	2527022.8530	981957.3370	5753940.7500
Hamina (Fin)	2795471.3860	1435427.7410	5531682.1280
Hanko (Fin)	2959173.1020	1254706.6550	5490604.5210
Helgoland (Ger)	3706045.0350	513713.1770	5148193.1900
Helsinki (Fin)	2885134.8290	1342693.6440	5509043.9240
Kemi (Fin)	2397170.2880	1093246.9200	5789077.1960
Klagshamn (Swe)	3527585.8260	807513.8050	5234549.4560
Klaipeda (Lit)	–	–	–
Kronstadt (Rus)	–	–	–
List/Sylt (Ger)	3625340.1590	5378854.8410	5202539.3040
Mäntyluoto (Fin)	2831096.8750	1113102.7630	5587164.9240
Molas (Lit)	–	–	–
Ölands N. U. (Swe)	3295551.6790	1012564.8840	5348113.5190
Raahe (Fin)	2492699.6820	1131503.6280	5741504.1800
Ratan (Swe)	2620087.7400	1000008.3720	5709322.5000
Shepelevo (Rus)	–	–	–
Spikarna (Swe)	2828573.5470	893623.7640	5627446.8610
Stockholm (Swe)	3101011.4900	1013009.1090	5462375.0730
Swinoujscie (Pol)	3649458.4740	927709.8850	5130741.4690
Ustka (Pol)	3545014.4560	1073939.6850	5174949.7730
Vaasa (Fin)	2691307.2790	1063691.5640	5664806.2010
Visby (Swe)	3249304.5320	1073624.8010	5364362.8610
Warnemünde (Ger)	3658230.7910	783507.3140	5148395.8730

Ger: Germany, Fin: Finland, Swe: Sweden, Lit: Lithuania, Pol: Poland, Rus: Russia

Table 2–2: Cartesian coordinates of GPS stations of BSL project second campaign 1993.4, in ITRF 93 reference frame.

Station Name	X (m)	Y (m)	Z (m)
Borkum (Ger)	3770668.1000	446076.4240	5107686.2360
Degerby (Fin)	2994005.4960	1112565.2460	5502270.9240
Furuögrund (Swe)	2527022.9170	981957.2520	5753940.9600
Hamina (Fin)	2795471.2440	1435427.7570	5531682.2020
Hanko (Fin)	2959172.9990	1254706.6680	5490604.6630
Helgoland (Ger)	3706044.9580	513713.1520	5148193.3740
Helsinki (Fin)	2885134.7810	1342693.6430	5509044.0630
Kemi (Fin)	2397170.2290	1093246.8740	5789077.1460
Klagshamn (Swe)	3527585.8220	807513.8260	5234549.6830
Klaipeda (Lit)	3353590.2730	1302062.9830	5249159.3680
Kronstadt (Rus)	–	–	–
List/Sylt (Ger)	3625340.0140	537853.8050	5202539.0420
Mäntyluoto (Fin)	2831096.7560	1113102.7150	5587165.0170
Molas (Lit)	3358793.4290	1294907.3540	5247584.3520
Ölands N. U. (Swe)	3295551.6450	1012564.8500	5348113.6740
Raahe (Fin)	2493889.7970	1131220.2470	5741045.9690
Ratan (Swe)	2620087.6710	1000008.2270	5709322.5780
Shepelevo (Rus)	2796394.3740	1556360.1010	5498639.2920
Spikarna (Swe)	2828573.5140	893623.6890	5627447.0490
Stockholm (Swe)	3101011.4990	1013009.1210	5462375.2820
Swinoujscie (Pol)	3649458.4600	927709.9170	5130741.6590
Ustka (Pol)	3545014.3980	1073939.7300	5174949.9340
Vaasa (Fin)	2691307.2900	1063691.4760	5664806.3350
Visby (Swe)	3249304.5220	1073624.8500	5364363.1060
Warnemünde (Ger)	3658217.6960	783004.6440	5148504.2800

Ger: Germany, Fin: Finland, Swe: Sweden, Lit: Lithuania, Pol: Poland, Rus: Russia

Table 2–3: Cartesian coordinates of GPS stations of BSL third campaign 1997.4, in ITRF 96 reference frame.

Station Name	X (m)	Y (m)	Z (m)
Borkum (Ger)	3770667.9990	446076.4900	5107686.2080
Degerby (Fin)	2994064.9360	1112559.0570	5502241.3760
Furuögrund (Swe)	2527022.8720	981957.2890	5753940.9920
Hamina (Fin)	2795471.2050	1435427.7920	5531682.2000
Hanko (Fin)	2959210.9710	1254679.1200	5490594.4410
Helgoland (Ger)	3706044.9440	513713.2150	5148193.4470
Helsinki (Fin)	2885137.3910	1342710.2300	5509039.1190
Kemi (Fin)	2397071.5770	1093330.3130	5789108.4470
Klagshamn (Swe)	3527585.7670	807513.8950	5234549.7020
Klaipeda (Lit)	3353590.2430	1302063.0140	5249159.4120
Kronstadt (Rus)	2776311.8190	1587590.1310	5499880.1330
List/Sylt (Ger)	3625339.9122	537853.8700	5202539.0260
Mäntyluoto (Fin)	2831096.7190	1113102.7640	5587165.0460
Molas (Lit)	3358793.3810	1294907.4050	5247584.4010

Tabelle 2–3: (continued)

Station Name	X (m)	Y (m)	Z (m)
Ölands N. U. (Swe)	3295551.5710	1012564.9060	5348113.6690
Raahe (Fin)	2494035.0240	1131370.9940	5740955.4100
Ratan (Swe)	2620087.6290	1000008.2700	5709322.6040
Shepelevo (Rus)	2796394.9140	1556363.7830	5498638.0600
Spikarna (Swe)	2828573.4640	893623.7290	5627447.0690
Stockholm (Swe)	3101008.8620	1013021.0370	5462373.3830
Swinoujscie (Pol)	3648326.5170	924984.0310	5132035.2720
Ustka (Pol)	3545014.3300	1073939.7720	5174949.9470
Vaasa (Fin)	2691307.2540	1063691.5240	5664806.3800
Visby (Swe)	3249304.4370	1073624.8910	5364363.0730
Warnemünde (Ger)	3658231.7070	783518.3220	5148404.3509

Ger: Germany, Fin: Finland, Swe: Sweden, Lit: Lithuania, Pol: Poland, Rus: Russia

Table 2–4: Jacobi ellipsoidal coordinates of the GPS stations of BSL project first campaign 1990.8, with respect to reference ellipsoid of WGD2000 in mean tide system, $a = (6378136.701 \pm 0.053)m$ and $b = (6356751.661 \pm 0.052)m$.

Station Name	λ °	'	"	ϕ °	'	"	u m
Borkum (Ger)	6	44	48.592	53	27	56.290	6356797.4401
Degerby (Fin)	20	23	5.799	59	56	54.812	6356772.7694
Furuögrund (Swe)	21	14	6.953	64	50	43.972	6356784.9708
Hamina (Fin)	27	10	47.061	60	28	56.235	6356769.0382
Hanko (Fin)	22	58	38.020	59	44	21.212	6356773.7771
Helgoland (Ger)	7	53	30.345	54	5	0.468	6356795.7627
Helsinki (Fin)	24	57	23.340	60	4	14.304	6356776.0350
Kemi (Fin)	24	30	56.527	65	36	5.357	6356772.7375
Klagshamn (Swe)	12	53	37.154	55	25	56.873	6356790.0414
Klaipeda (Lit)	–	–	–	–	–	–	–
Kronstadt (Rus)	–	–	–	–	–	–	–
List/Sylt (Ger)	56	1	12.319	38	47	11.504	8305186.1308
Mäntyluoto (Fin)	21	27	47.774	61	30	49.223	6356773.4710
Molas (Lit)	–	–	–	–	–	–	–
Ölands N. U. (Swe)	17	4	46.851	57	16	48.643	6356783.6250
Raahe (Fin)	24	24	52.486	64	35	0.055	6356772.4568
Ratan (Swe)	20	53	25.243	63	54	56.307	6356775.1165
Shepelevo (Rus)	–	–	–	–	–	–	–
Spikarna (Swe)	17	31	57.907	62	17	3.822	6356779.3644
Stockholm (Swe)	18	5	26.484	59	14	16.253	6356788.2619
Swinoujscie (Pol)	14	15	45.951	53	48	58.479	6356790.0989
Ustka (Pol)	16	51	13.868	54	29	48.357	6356786.0870
Vaasa (Fin)	21	33	55.917	63	1	3.025	6356771.3215
Visby (Swe)	18	17	3.922	57	33	7.925	6356779.3580
Warnemünde (Ger)	12	5	19.600	54	5	11.799	6356793.1157

Table 2–5: Jacobi ellipsoidal coordinates of the GPS stations of BSL project, second campaign 1993.4, with respect to reference ellipsoid of WGD2000 in mean tide system, $a = (6378136.701 \pm 0.053)m$ $b = (6356751.661 \pm 0.052)m$.

Station Name	λ			ϕ			u
	°	'	"	°	'	"	m
Borkum (Ger)	6	44	48.587	53	27	56.294	6356797.0766
Degerby (Fin)	20	23	5.802	59	56	54.816	6356772.8204
Furuögrund (Swe)	21	14	6.945	64	50	43.974	6356785.1749
Hamina (Fin)	27	10	47.066	60	28	56.239	6356769.0445
Hanko (Fin)	22	58	38.023	59	44	21.217	6356773.8537
Helgoland (Ger)	7	53	30.344	54	5	0.474	6356795.8647
Helsinki (Fin)	24	57	23.341	60	4	14.307	6356776.1370
Kemi (Fin)	24	30	56.525	65	36	5.358	6356772.6609
Klagshamn (Swe)	12	53	37.155	55	25	56.877	6356790.2264
Klaipeda (Lit)	21	13	9.013	55	39	54.157	6356805.2088
Kronstadt (Rus)	–	–	–	–	–	–	–
List/Sylt (Ger)	8	26	19.755	54	55	37.511	6356797.0447
Mäntyluoto (Fin)	21	27	47.774	61	30	49.228	6356773.4901
Molas (Lit)	21	4	58.889	55	38	24.697	6356781.6159
Ölands N. U. (Swe)	17	4	46.850	57	16	48.646	6356783.7334
Raahe (Fin)	24	23	55.998	64	34	25.438	6356772.1890
Ratan (Swe)	20	53	25.235	63	54	56.312	6356775.1357
Shepelevo (Rus)	29	5	54.760	59	52	59.790	6356771.8318
Spikarna (Swe)	17	31	57.902	62	17	3.826	6356779.5047
Stockholm (Swe)	18	5	26.485	59	14	16.256	6356788.4469
Swinoujscie (Pol)	14	15	45.953	53	48	58.483	6356790.2519
Ustka (Pol)	16	51	13.872	54	29	48.361	6356786.1954
Vaasa (Fin)	21	33	55.911	63	1	3.028	6356771.4300
Visby (Swe)	18	17	3.925	57	33	7.929	6356779.5685
Warnemünde (Ger)	12	4	52.651	54	5	16.952	6356811.9505

Table 2–6: Jacobi ellipsoidal coordinates of the GPS stations of BSL project, third campaign 1997.4, with respect to reference ellipsoid of WGD2000 in mean tide system, $a = (6378136.701 \pm 0.053)m$ $b = (6356751.661 \pm 0.052)m$.

Station Name	λ			ϕ			u
	°	'	"	°	'	"	m
Borkum (Ger)	6	44	48.591	53	27	56.296	6356797.0000
Degerby (Fin)	20	23	4.091	59	56	52.837	6356773.9748
Furuögrund (Swe)	21	14	6.949	64	50	43.975	6356785.1940
Hamina (Fin)	27	10	47.069	60	28	56.240	6356769.0318
Hanko (Fin)	22	58	35.444	59	44	20.373	6356777.1894
Helgoland (Ger)	7	53	30.348	54	5	0.476	6356795.9221
Helsinki (Fin)	24	57	24.245	60	4	13.965	6356776.5070
Kemi (Fin)	24	31	5.674	65	36	7.401	6356778.4778
Klagshamn (Swe)	12	53	37.160	55	25	56.878	6356790.2200
Klaipeda (Lit)	21	13	9.016	55	39	54.158	6356805.2343
Kronstadt (Rus)	29	45	44.638	59	54	20.012	6356772.6865
List/Sylt (Ger)	8	26	19.759	54	55	37.513	6356796.9809
Mäntyluoto (Fin)	21	27	47.778	61	30	49.229	6356773.5092

Tabelle 2–6: (continued)

Station Name	λ			ϕ			u
	°	'	''	°	'	''	m
Molas (Lit)	21	4	58.893	55	38	24.698	6356781.6414
Ölands N. U. (Swe)	17	4	46.854	57	16	48.648	6356783.6952
Raahe (Fin)	24	24	1.820	64	34	18.495	6356773.6451
Ratan (Swe)	20	53	25.239	63	54	56.313	102253.32707
Shepelevo (Rus)	29	5	54.951	59	52	59.707	6356771.9019
Spikarna (Swe)	17	31	57.906	62	17	3.827	6356779.5047
Stockholm (Swe)	18	5	27.253	59	14	16.192	6356787.4200
Swinoujscie (Pol)	14	13	36.455	53	50	9.408	6356794.5253
Ustka (Pol)	16	51	13.875	54	29	48.363	6356786.1763
Vaasa (Fin)	21	33	55.915	63	1	3.029	6356771.4619
Visby (Swe)	18	17	3.929	57	33	7.931	6356779.5047
Warnemünde (Ger)	12	5	20.183	54	5	11.875	6356801.8730

Table 2–7: Orthometric height of the GPS stations of BSL project, first campaign 1990.8, derived from different sources (case1–case5).

Station Name	orthometric height (case 1) (*M. Poutanen et al.*, 1999)	orthometric height (case 2) (*J. Kakkuri*, 1995)	orthometric height (case 3) (*J. Kakkuri and M. Poutanen*, 1997)	orthometric height (case 4) (*M. Poutanen et al.*, 1999)	orthometric height (case 5) (*EGG97*)
Borkum (Ger)	4.581	5.454	–	5.884	5.556
Degerby (Fin)	1.695	2.345	2.428	2.541	2.422
Furuögrund (Swe)	10.912	11.597	11.541	11.830	12.180
Hamina (Fin)	1.619	2.374	2.357	2.645	2.458
Hanko (Fin)	1.762	2.434	2.473	2.694	2.626
Helgoland (Ger)	4.539	–	–	5.196	4.997
Helsinki (Fin)	6.033	6.654	6.663	6.933	6.820
Kemi (Fin)	1.246	2.161	2.248	2.546	2.485
Klagshamn (Swe)	2.039	2.551	2.169	2.775	2.759
Klaipeda (Lit)	–	–	–	–	–
Kronstadt (Rus)	–	–	–	–	–
List/Sylt (Ger)	4.160	4.904	–	5.302	5.135
Mäntyluoto (Fin)	2.467	3.208	3.222	3.451	–
Molas (Lit)	–	–	–	–	–
Ölands N. U. (Swe)	4.118	4.493	4.714	4.961	4.729
Raahe (Fin)	2.289	3.127	3.168	3.510	3.411
Ratan (Swe)	1.476	2.361	2.268	2.581	2.633
Shepelevo (Rus)	–	–	–	–	–
Spikarna (Swe)	1.872	2.938	2.603	2.809	2.943
Stockholm (Swe)	12.865	13.532	13.620	13.729	13.699
Swinoujscie (Pol)	2.312	2.877	2.516	2.947	3.004
Ustka (Pol)	1.535	–	1.720	2.180	2.379
Vaasa (Fin)	1.128	1.828	1.772	2.131	2.060
Visby (Swe)	1.986	2.278	2.547	2.713	2.533
Warnemünde (Ger)	–	–	2.963	3.307	3.246

Table 2–8: Orthometric height of the GPS stations of BSL project, second campaign 1993.4, derived from different sources (case1–case5).

Station Name	orthometric height (case 1) (*M. Poutanen et al.*, 1999)	orthometric height (case 2) (*J. Kakkuri*, 1995)	orthometric height (case 3) (*J. Kakkuri and M. Poutanen*, 1997)	orthometric height (case 4) (*M. Poutanen et al.*, 1999)	or-thometric height (case 5) (*EGG97*)
Borkum (Ger)	4.578	4.433	–	4.740	4.874
Degerby (Fin)	1.681	1.708	1.791	1.745	1.906
Furuögrund (Swe)	10.936	11.074	11.018	11.121	11.308
Hamina (Fin)	1.624	1.710	1.693	1.818	1.982
Hanko (Fin)	1.769	1.824	1.863	1.926	2.086
Helgoland (Ger)	4.537	–	–	4.562	4.687
Helsinki (Fin)	6.039	6.069	6.078	6.187	6.349
Kemi (Fin)	1.266	1.355	1.442	1.562	1.742
Klagshamn (Swe)	2.039	2.060	1.678	2.152	2.286
Klaipeda (Lit)	28.209	–	–	28.323	–
Kronstadt (Rus)	–	–	–	–	–
List/Sylt (Ger)	4.159	3.930	–	4.199	4.330
Mäntyluoto (Fin)	2.484	2.533	2.547	2.608	–
Molas (Lit)	4.577	–	–	4.664	4.799
Ölands N. U. (Swe)	4.122	3.906	4.127	4.230	4.375
Raahe (Fin)	2.086	2.139	2.180	2.337	2.523
Ratan (Swe)	1.500	1.660	1.567	1.698	1.881
Shepelevo (Rus)	–	4.211	4.262	4.317	–
Spikarna (Swe)	1.893	2.365	2.030	2.063	2.237
Stockholm (Swe)	12.873	13.020	13.108	13.062	13.219
Swinoujscie (Pol)	2.309	2.367	2.006	2.315	2.439
Ustka (Pol)	1.532	–	1.187	1.521	1.649
Vaasa (Fin)	1.149	1.227	1.171	1.353	1.551
Visby (Swe)	1.986	1.775	2.044	2.065	2.212
Warnemünde (Ger)	21.291	–	21.146	21.367	21.492

Table 2–9: Orthometric height of the GPS stations of BSL project, 3rd campaign 1997.4, derived from different sources (case1–case5).

Station Name	orthometric height (case 1) (*M. Poutanen et al.*, 1999)	orthometric height (case 2) (*J. Kakkuri*, 1995)	orthometric height (case 3) (*J. Kakkuri and M. Poutanen*, 1997)	orthometric height (case 4) (*M. Poutanen et al.*, 1999)	orthometric height (case 5) (*Kakkuri*, 2000)
Borkum (Ger)	4.574	4.405	–	4.712	4.846
Degerby (Fin)	2.825	2.877	2.960	2.914	3.075
Furuögrund (Swe)	10.972	11.108	11.052	11.155	11.342
Hamina (Fin)	1.631	1.697	1.680	1.805	1.969
Hanko (Fin)	5.118	5.173	5.212	5.275	5.435
Helgoland (Ger)	4.531	–	–	4.610	4.735
Helsinki (Fin)	6.420	6.455	6.464	6.573	6.735
Kemi (Fin)	7.092	7.185	7.272	7.392	7.572

Tabelle 2–9: (continued)

Station Name	orthometric height (case 1) (*M. Poutanen et al.*, 1999)	orthometric height (case 2) (*J. Kakkuri*, 1995)	orthometric height (case 3) (*J. Kakkuri and M. Poutanen*, 1997)	orthometric height (case 4) (*M. Poutanen et al.*, 1999)	orthometric height (case 5) (*Kakkuri*, 2000)
Klagshamn (Swe)	2.038	2.099	1.717	2.191	2.325
Klaipeda (Lit)	28.209	–	–	28.336	–
Kronstadt (Rus)	–	–	–	4.739	–
List/Sylt (Ger)	4.155	3.916	–	4.185	4.316
Mäntyluoto (Fin)	2.509	2.562	2.576	2.637	–
Molas (Lit)	4.577	–	–	4.679	4.814
Ölands N. U. (Swe)	4.127	3.917	4.138	4.241	4.386
Raahe (Fin)	3.528	3.607	3.648	3.805	3.991
Ratan (Swe)	1.535	1.691	1.598	1.729	1.912
Shepelevo (Rus)	–	4.282	4.333	4.388	–
Spikarna (Swe)	1.924	2.390	2.055	2.088	2.262
Stockholm (Swe)	11.905	12.027	12.115	12.069	12.226
Swinoujscie (Pol)	–	6.681	6.320	6.629	6.753
Ustka (Pol)	1.528	–	1.181	1.515	1.643
Vaasa (Fin)	1.180	1.275	1.219	1.401	1.599
Visby (Swe)	1.992	1.771	2.040	2.061	2.208
Warnemünde Ger)	11.319	–	11.089	11.310	11.435

3. Baltic Sea Level Project First Campaign 1990.8

Here we present the results of w_0 computation based on the GPS observations of the Baltic Sea Level project, 1st campaign, in five cases introduced before. *Table 3–1–Table 3–5* are presenting the gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u)$ computed based on the orthometric heights of case 1–case 5. A summary of the mean values of w_0 derived from different cases and the standard deviation of those mean values is given in *Table 3–6*.

Table 3–1: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u)$ computed based on the orthometric heights of the case 1 (*M. Poutanen et al.*, 1999), and gauge value of geoid potential W_0 for the epoch of 1990.8. The W_0 at *Lisk* (Ger) found as outlier, not included in the mean.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\Delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598475.745	38330.956	44.956	62636851.657
Degerby (Fin)	62609712.120	27124.194	16.643	62636852.957
Furuögrund (Swe)	62617207.876	19542.016	107.183	62636857.076
Hamina (Fin)	62610584.084	26255.428	15.897	62636855.409
Hanko (Fin)	62609368.721	27467.501	17.301	62636853.523

Tabelle 3–1: (continued)

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\Delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Helgoland (Ger)	62599594.591	37218.925	44.546	62636858.063
Helsinki (Fin)	62609873.004	26924.677	59.239	62636856.920
Kemi (Fin)	62618386.248	18455.996	12.240	62636854.484
Klagshamn (Swe)	62602016.194	34818.793	20.014	62636855.001
Klaipeda (Lit)	–	–	–	–
Kronstadt (Rus)	–	–	–	–
List/Sylt (Ger)	47933936.649	111867.236	23.866	48045827.752
Mäntyluoto (Fin)	62612227.796	24604.331	24.226	62636856.353
Molas (Lit)	–	–	–	–
Ölands N. U. (Swe)	62605215.375	31601.731	40.426	62636857.531
Raahe (Fin)	62616906.611	19924.257	22.483	62636853.351
Ratan (Swe)	62615931.246	20910.717	14.497	62636856.460
Shepelevo (Rus)	–	–	–	–
Spikarna (Swe)	62613442.333	23395.357	18.384	62636856.074
Stockholm (Swe)	62608433.790	28295.608	126.314	62636855.712
Swinoujscie (Pol)	62599134.600	37698.887	22.690	62636856.177
Ustka (Pol)	62600357.058	36479.270	15.066	62636851.393
Vaasa (Fin)	62614579.338	22265.876	11.078	62636856.292
Visby (Swe)	62605700.835	31135.657	19.497	62636855.989
Warnemünde (Ger)	62599616.182	37213.250	–	–
			mean (m^2/s^2)	62636855.285
			std (m^2/s^2)	0.4268

Table 3–2: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u)$ computed based on the orthometric heights of the case 2 (*J. Kakkuri*, 1995), and gauge value of geoid potential W_0 for the epoch of 1990.8. The W_0 at *Lisk* (Ger) found as outlier, and not included in the mean.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\Delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598475.745	38330.956	47.527	62636854.228
Degerby (Fin)	62609712.120	27124.194	16.436	62636852.751
Furuögrund (Swe)	62617207.876	19542.016	107.999	62636857.891
Hamina (Fin)	62610584.084	26255.428	16.595	62636856.106
Hanko (Fin)	62609368.721	27467.501	17.625	62636853.847
Helgoland (Ger)	62599594.591	37218.925	–	–
Helsinki (Fin)	62609873.004	26924.677	59.082	62636856.763
Kemi (Fin)	62618386.248	18455.996	12.780	62636855.024
Klagshamn (Swe)	62602016.194	34818.793	19.572	62636854.559
Klaipeda (Lit)	–	–	–	–
Kronstadt (Rus)	–	–	–	–
List/Sylt (Ger)	47933936.649	111867.236	22.926	48045826.811
Mäntyluoto (Fin)	62612227.796	24604.331	24.746	62636856.873
Molas (Lit)	–	–	–	–
Ölands N. U. (Swe)	62605215.375	31601.731	37.795	62636854.900

Tabelle 3–2: (continued)

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\Delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Raahe (Fin)	62616906.611	19924.257	23.858	62636854.726
Ratan (Swe)	62615931.246	20910.717	17.257	62636859.220
Shepelevo (Rus)	–	–	–	–
Spikarna (Swe)	62613442.333	23395.357	22.784	62636860.473
Stockholm (Swe)	62608433.790	28295.608	126.608	62636856.006
Swinoujscie (Pol)	62599134.600	37698.887	22.406	62636855.893
Ustka (Pol)	62600357.058	36479.270	–	–
Vaasa (Fin)	62614579.338	22265.876	11.500	62636856.714
Visby (Swe)	62605700.835	31135.657	16.326	62636852.818
Warnemünde (Ger)	62599616.182	37213.250	–	–
			mean (m^2/s^2)	62636855.811
			std (m^2/s^2)	0.5056

Table 3–3: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u)$ computed based on orthometric heights of the case 3 (*J. Kakkuri and M. Poutanen*, 1997), and gauge value of geoid potential W_0 for the epoch of 1990.8. The W_0 at *Ustka* (Pol) found as outlier, and not included in the mean.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\Delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598475.745	38330.956	–	–
Degerby (Fin)	62609712.120	27124.194	17.251	62636853.566
Furuögrund (Swe)	62617207.876	19542.016	107.449	62636857.341
Hamina (Fin)	62610584.084	26255.428	16.428	62636855.939
Hanko (Fin)	62609368.721	27467.501	18.008	62636854.230
Helgoland (Ger)	62599594.591	37218.925	–	–
Helsinki (Fin)	62609873.004	26924.677	59.170	62636856.852
Kemi (Fin)	62618386.248	18455.996	13.634	62636855.879
Klagshamn (Swe)	62602016.194	34818.793	15.822	62636850.809
Klaipeda (Lit)	–	–	–	–
Kronstadt (Rus)	–	–	–	–
List/Sylt (Ger)	47933936.649	111867.236	–	–
Mäntyluoto (Fin)	62612227.796	24604.331	24.884	62636857.010
Molas (Lit)	–	–	–	–
Ölands N. U. (Swe)	62605215.375	31601.731	39.964	62636857.070
Raahe (Fin)	62616906.611	19924.257	24.261	62636855.129
Ratan (Swe)	62615931.246	20910.717	16.344	62636858.307
Shepelevo (Rus)	–	–	–	–
Spikarna (Swe)	62613442.333	23395.357	19.494	62636857.184
Stockholm (Swe)	62608433.790	28295.608	127.472	62636856.870
Swinoujscie (Pol)	62599134.600	37698.887	18.863	62636852.350
Ustka (Pol)	62600357.058	36479.270	11.061	62636847.389

Tabelle 3–3: (continued)

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\Delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Vaasa (Fin)	62614579.338	22265.876	10.950	62636856.164
Visby (Swe)	62605700.835	31135.657	18.967	62636855.459
Warnemünde (Ger)	62599616.182	37213.250	22.367	62636851.798
			mean (m^2 / s^2)	62636855.409
			std (m^2 / s^2)	0.5227

Table 3–4: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u)$ computed based on orthometric heights of the case 4 (*M. Poutanen et al.* 1999), and gauge value of geoid potential W_0 for the epoch of 1990.8. The W_0 at *Ustka* (Pol) and *Lisk* (Ger) are outliers and excluded from mean.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\Delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598475.745	38330.956	50.540	62636857.241
Degerby (Fin)	62609712.120	27124.194	16.800	62636853.114
Furuögrund (Swe)	62617207.876	19542.016	108.460	62636858.352
Hamina (Fin)	62610584.084	26255.428	17.655	62636857.167
Hanko (Fin)	62609368.721	27467.501	18.627	62636854.849
Helgoland (Ger)	62599594.591	37218.925	44.782	62636858.298
Helsinki (Fin)	62609873.004	26924.677	60.241	62636857.922
Kemi (Fin)	62618386.248	18455.996	14.813	62636857.058
Klagshamn (Swe)	62602016.194	34818.793	20.475	62636855.462
Klaipeda (Lit)	–	–	–	–
Kronstadt (Rus)	–	–	–	–
List/Sylt (Ger)	47933936.649	111867.236	24.469	48045828.354
Mäntyluoto (Fin)	62612227.796	24604.331	25.483	62636857.609
Molas (Lit)	–	–	–	–
Ölands N. U. (Swe)	62605215.375	31601.731	40.975	62636858.081
Raahe (Fin)	62616906.611	19924.257	25.803	62636856.671
Ratan (Swe)	62615931.246	20910.717	17.631	62636859.594
Shepelevo (Rus)	–	–	–	–
Spikarna (Swe)	62613442.333	23395.357	19.818	62636857.508
Stockholm (Swe)	62608433.790	28295.608	127.021	62636856.419
Swinoujscie (Pol)	62599134.600	37698.887	21.895	62636855.382
Ustka (Pol)	62600357.058	36479.270	14.339	62636850.667
Vaasa (Fin)	62614579.338	22265.876	12.738	62636857.951
Visby (Swe)	62605700.835	31135.657	19.173	62636855.665
Warnemünde (Ger)	62599616.182	37213.250	24.536	62636853.967
			mean (m^2 / s^2)	62636856.753
			std (m^2 / s^2)	0.3780

Table 3–5: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u)$ based on orthometric heights of the case 5 (*J. Kakkuri*, 2000), and gauge value of geoid potential W_0 for the epoch of 1990.8. The W_0 at *Ustka* (Pol) and *Lisk* (Ger) are outliers and not included in the mean value computation.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\Delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598475.745	38330.956	51.855	62636858.556
Degerby (Fin)	62609712.120	27124.194	18.380	62636854.695
Furuögrund (Swe)	62617207.876	19542.016	110.297	62636860.189
Hamina (Fin)	62610584.084	26255.428	19.265	62636858.777
Hanko (Fin)	62609368.721	27467.501	20.198	62636856.420
Helgoland (Ger)	62599594.591	37218.925	46.008	62636859.525
Helsinki (Fin)	62609873.004	26924.677	61.831	62636859.513
Kemi (Fin)	62618386.248	18455.996	16.581	62636858.826
Klagshamn (Swe)	62602016.194	34818.793	21.790	62636856.777
Klaipeda (Lit)	–	–	–	–
Kronstadt (Rus)	–	–	–	–
List/Sylt (Ger)	47933936.649	111867.236	25.220	48045829.106
Mäntyluoto (Fin)	62612227.796	24604.331	–	–
Molas (Lit)	–	–	–	–
Ölands N. U. (Swe)	62605215.375	31601.731	42.399	62636859.504
Raahe (Fin)	62616906.611	19924.257	27.630	62636858.498
Ratan (Swe)	62615931.246	20910.717	19.428	62636861.391
Shepelevo (Rus)	–	–	–	–
Spikarna (Swe)	62613442.333	23395.357	21.527	62636859.216
Stockholm (Swe)	62608433.790	28295.608	128.562	62636857.960
Swinoujscie (Pol)	62599134.600	37698.887	23.112	62636856.599
Ustka (Pol)	62600357.058	36479.270	15.596	62636851.923
Vaasa (Fin)	62614579.338	22265.876	14.682	62636859.896
Visby (Swe)	62605700.835	31135.657	20.616	62636857.108
Warnemünde (Ger)	62599616.182	37213.250	25.763	62636855.194
			mean (m^2/s^2)	62636858.258
			std (m^2/s^2)	0.4218

Now let us summarise those results we obtained in five different cases for the geoid potential value w_0, based on GPS observations of Baltic Sea level project, 1[st] campaign, epoch 1990.8, in *Table 3–6*. From a review of *Table 3–6* following conclusions can be made:

(i) The most consistent results are corresponding to case 4.

(ii) The results of case 5 show a shift of about 2.4 (m^2/s^2) with respect to mean value of w_0, calculated from other four cases.

(iii) The weighted mean of the geoid potential value w_0 based on the GPS observations of Baltic Sea Level project, 1[st] campaign and orthometric heights of all cases is (62636856.434 $\pm$ 0.558) (m^2/s^2) and based on the orthometric heights of case1–case4 is 62636855.922 $\pm$ 0.366 (m^2/s^2).

Table 3–6: Summary of w_0 values computed from the GPS observations of the Baltic Sea level project, first campaign 1990.8, for the case1–case5.

case	w_0 (m^2/s^2)	$std(w_0)$ (m^2/s^2)
case 1	62636855.285	0.4268
case 2	62636855.811	0.5056
case 3	62636855.409	0.5227
case 4	62636856.753	0.3780
case 5	62636858.258	0.4218

4. Baltic Sea Level Project, Second Campaign 1993.4

In this section, we will review the results of w_0 computation based on the GPS observations of Baltic Sea level project, 2[nd] campaign, in five cases introduced before. *Table 4–1–Table 4–5* are presenting the gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u)$ computed based on the orthometric heights of case 1–case 5. A summary of the mean values derived from different cases and the standard deviation of those mean values is given in *Table 4–6*.

Table 4–1: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u;H_0)$, computed based on the orthometric heights of the case 1 (*M. Poutanen et al.*, 1999), and gauge value of geoid potential W_0 for the epoch of 1993.4.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598479.287	38330.949	44.927	62636855.163
Degerby (Fin)	62609711.662	27124.193	16.505	62636852.360
Furuögrund (Swe)	62617205.887	19542.017	107.419	62636855.323
Hamina (Fin)	62610584.027	26255.426	15.946	62636855.400
Hanko (Fin)	62609367.961	27467.499	17.370	62636852.829
Helgoland (Ger)	62599593.587	37218.924	44.526	62636857.037
Helsinki (Fin)	62609872.036	26924.677	59.298	62636856.010
Kemi (Fin)	62618386.991	18455.995	12.436	62636855.423
Klagshamn (Swe)	62602014.353	34818.793	20.014	62636853.160
Klaipeda (Lit)	62602168.578	34409.334	276.885	62636854.797
Kronstadt (Rus)	–	–	–	–
List/Sylt (Ger)	62601101.969	35713.293	40.821	62636856.083
Mäntyluoto (Fin)	62612227.594	24604.328	24.393	62636856.316
Molas (Lit)	62602356.820	34452.784	44.926	62636854.530
Ölands N. U. (Swe)	62605214.319	31601.730	40.465	62636856.514
Raahe (Fin)	62616895.232	19938.331	20.489	62636854.053
Ratan (Swe)	62615931.059	20910.716	14.733	62636856.508
Shepelevo (Rus)	62609585.155	27231.095	–	–
Spikarna (Swe)	62613440.944	23395.356	18.591	62636854.891
Stockholm (Swe)	62608431.963	28295.609	126.392	62636853.964

Tabelle 4–1: (continued)

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Swinoujscie (Pol)	62599133.126	37698.887	22.661	62636854.674
Ustka (Pol)	62600356.012	36479.269	15.036	62636850.317
Vaasa (Fin)	62614578.263	22265.875	.284	62636855.423
Visby (Swe)	62605698.773	31135.657	19.497	62636853.927
Warnemünde (Ger)	62599433.944	37210.901	208.957	62636853.802
			mean (m^2s^{-2})	62636854.718
			std (m^2s^{-2})	0.3248

Table 4–2: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u;H_0)$ computed based on orthometric heights of the case 2 (*J. Kakkuri*, 1995) and gauge value of geoid potential W_0 for the epoch of 1993.4.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598479.287	38330.949	43.504	62636853.740
Degerby (Fin)	62609711.662	27124.193	16.770	62636852.625
Furuögrund (Swe)	62617205.887	19542.017	108.775	62636856.678
Hamina (Fin)	62610584.027	26255.426	16.791	62636856.244
Hanko (Fin)	62609367.961	27467.499	17.910	62636853.369
Helgoland (Ger)	62599593.587	37218.924	–	–
Helsinki (Fin)	62609872.036	26924.677	59.592	62636856.305
Kemi (Fin)	62618386.991	18455.995	13.310	62636856.297
Klagshamn (Swe)	62602014.353	34818.793	20.220	62636853.366
Klaipeda (Lit)	62602168.578	34409.334	–	–
Kronstadt (Rus)	–	–	–	–
List/Sylt (Ger)	62601101.969	35713.293	38.574	62636853.836
Mäntyluoto (Fin)	62612227.594	24604.328	24.874	62636856.797
Molas (Lit)	62602356.820	34452.784	–	–
Ölands N. U. (Swe)	62605214.319	31601.730	38.345	62636854.393
Raahe (Fin)	62616895.232	19938.331	21.010	62636854.573
Ratan (Swe)	62615931.059	20910.716	16.305	62636858.079
Shepelevo (Rus)	62609585.155	27231.095	41.348	62636857.598
Spikarna (Swe)	62613440.944	23395.356	23.226	62636859.526
Stockholm (Swe)	62608431.963	28295.609	127.836	62636855.407
Swinoujscie (Pol)	62599133.126	37698.887	23.230	62636855.243
Ustka (Pol)	62600356.012	36479.269	–	–
Vaasa (Fin)	62614578.263	22265.875	12.050	62636856.189
Visby (Swe)	62605698.773	31135.657	17.426	62636851.855
Warnemünde (Ger)	62599433.944	37210.901	–	–
			mean (m^2s^{-2})	62636855.375
			std (m^2s^{-2})	0.4557

Table 4–3: Gravitational potential $U(\lambda,\phi,\eta)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u;H_0)$ computed based on orthometric heights of the case 3 (*J. Kakkuri and M. Poutanen*, 1997), and gauge value of geoid potential W_0 for the epoch of 1993.4. The W_0 value at *Ustka* (Pol) and Klagshamn (Swe) are outliers and removed from the mean.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598479.287	38330.949	–	–
Degerby (Fin)	62609711.662	27124.193	17.585	62636853.440
Furuögrund (Swe)	62617205.887	19542.017	108.225	62636856.128
Hamina (Fin)	62610584.027	26255.426	16.624	62636856.077
Hanko (Fin)	62609367.961	27467.499	18.293	62636853.752
Helgoland (Ger)	62599593.587	37218.924	–	–
Helsinki (Fin)	62609872.036	26924.677	59.681	62636856.393
Kemi (Fin)	62618386.991	18455.995	14.165	62636857.151
Klagshamn (Swe)	62602014.353	34818.793	16.470	62636849.616
Klaipeda (Lit)	62602168.578	34409.334	–	–
Kronstadt (Rus)	–	–	–	–
List/Sylt (Ger)	62601101.969	35713.293	–	–
Mäntyluoto (Fin)	62612227.594	24604.328	25.012	62636856.934
Molas (Lit)	62602356.820	34452.784	–	–
Ölands N. U. (Swe)	62605214.319	31601.730	40.514	62636856.563
Raahe (Fin)	62616895.232	19938.331	21.413	62636854.976
Ratan (Swe)	62615931.059	20910.716	15.391	62636857.166
Shepelevo (Rus)	62609585.155	27231.095	41.848	62636858.099
Spikarna (Swe)	62613440.944	23395.356	19.936	62636856.236
Stockholm (Swe)	62608431.963	28295.609	128.700	62636856.271
Swinoujscie (Pol)	62599133.126	37698.887	19.687	62636851.700
Ustka (Pol)	62600356.012	36479.269	11.650	62636846.931
Vaasa (Fin)	62614578.263	22265.875	11.500	62636855.639
Visby (Swe)	62605698.773	31135.657	20.066	62636854.496
Warnemünde (Ger)	62599433.944	37210.901	207.533	62636852.379
			mean (m^2s^{-2})	62636855.494
			std (m^2s^{-2})	0.4325

Table 4–4: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u;H_0)$, based on orthometric heights of the case 4 (*M. Poutanen et al.* 1999), and gauge value of geoid potential W_0 for the epoch of 1993.4. The W_0 at *Ustka* (Pol) is outlier and removed from the mean.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598479.287	38330.949	46.517	62636856.753
Degerby (Fin)	62609711.662	27124.193	17.134	62636852.988
Furuögrund (Swe)	62617205.887	19542.017	109.236	62636857.140
Hamina (Fin)	62610584.027	26255.426	17.851	62636857.305
Hanko (Fin)	62609367.961	27467.499	18.911	62636854.371

Tabelle 4–4: (continued)

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Helgoland (Ger)	62599593.587	37218.924	44.772	62636857.282
Helsinki (Fin)	62609872.036	26924.677	60.751	62636857.463
Kemi (Fin)	62618386.991	18455.995	15.344	62636858.330
Klagshamn (Swe)	62602014.353	34818.793	21.123	62636854.269
Klaipeda (Lit)	62602168.578	34409.334	278.004	62636855.916
Kronstadt (Rus)	–	–	–	–
List/Sylt (Ger)	62601101.969	35713.293	41.214	62636856.476
Mäntyluoto (Fin)	62612227.594	24604.328	25.611	62636857.533
Molas (Lit)	62602356.820	34452.784	45.779	62636855.384
Ölands N. U. (Swe)	62605214.319	31601.730	41.525	62636857.574
Raahe (Fin)	62616895.232	19938.331	22.955	62636856.518
Ratan (Swe)	62615931.059	20910.716	16.678	62636858.453
Shepelevo (Rus)	62609585.155	27231.095	42.389	62636858.639
Spikarna (Swe)	62613440.944	23395.356	20.260	62636856.560
Stockholm (Swe)	62608431.963	28295.609	128.248	62636855.820
Swinoujscie (Pol)	62599133.126	37698.887	22.720	62636854.733
Ustka (Pol)	62600356.012	36479.269	14.928	62636850.210
Vaasa (Fin)	62614578.263	22265.875	13.288	62636857.426
Visby (Swe)	62605698.773	31135.657	20.273	62636854.702
Warnemünde (Ger)	62599433.944	37210.901	209.702	62636854.548
			mean (m^2s^{-2})	62636856.356
			std (m^2s^{-2})	0.3177

Table 4–5: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u;H_0)$, computed based on orthometric heights of the case 5 (*J. Kakkuri*, 2000), and gauge value of geoid potential W_0 for the epoch of 1993.4. The W_0 at *Ustka* (Pol) is outlier and not included in the mean.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598479.287	38330.949	47.832	62636858.068
Degerby (Fin)	62609711.662	27124.193	18.714	62636854.569
Furuögrund (Swe)	62617205.887	19542.017	111.073	62636858.977
Hamina (Fin)	62610584.027	26255.426	19.462	62636858.915
Hanko (Fin)	62609367.961	27467.499	20.482	62636855.942
Helgoland (Ger)	62599593.587	37218.924	45.999	62636858.509
Helsinki (Fin)	62609872.036	26924.677	62.342	62636859.054
Kemi (Fin)	62618386.991	18455.995	17.112	62636860.098
Klagshamn (Swe)	62602014.353	34818.793	22.438	62636855.584
Klaipeda (Lit)	62602168.578	34409.334	–	–
Kronstadt (Rus)	–	–	–	–
List/Sylt (Ger)	62601101.969	35713.293	42.500	62636857.762
Mäntyluoto (Fin)	62612227.594	24604.328	–	–

Tabelle 4–5: (continued)

Station Name	Gravitational Potential $U(\lambda, \phi, u)$ $(m^2 s^{-2})$	Centrifugal Potential $V(\lambda, \phi, u)$ $(m^2 s^{-2})$	Free Air Reduction $\delta W(\lambda, \phi, u)$ $(m^2 s^{-2})$	Gauge Value W_0 $(m^2 s^{-2})$
Molas (Lit)	62602356.820	34452.784	47.105	62636856.709
Ölands N. U. (Swe)	62605214.319	31601.730	42.949	62636858.998
Raahe (Fin)	62616895.232	19938.331	24.782	62636858.345
Ratan (Swe)	62615931.059	20910.716	18.475	62636860.250
Shepelevo (Rus)	62609585.155	27231.095	–	–
Spikarna (Swe)	62613440.944	23395.356	21.969	62636858.269
Stockholm (Swe)	62608431.963	28295.609	129.789	62636857.361
Swinoujscie (Pol)	62599133.126	37698.887	23.936	62636855.950
Ustka (Pol)	62600356.012	36479.269	16.184	62636851.466
Vaasa (Fin)	62614578.263	22265.875	15.232	62636859.371
Visby (Swe)	62605698.773	31135.657	21.716	62636856.145
Warnemünde (Ger)	62599433.944	37210.901	210.929	62636855.775
			mean $(m^2 s^{-2})$	62636857.733
			std $(m^2 s^{-2})$	0.3658

Now let us summarise those results obtained for w_0 in five different cases, based on GPS observation of Baltic Sea level project, 2nd campaign, epoch 1993.4, in *Table 4–6*. From a review of *Table 4–6* following conclusions can be made:

(i) The results of second campaign are in general more accurate than the first campaign, which is due to the more accurate GPS observation, made under more favourable ionspheric condition as is explained by *J. Kakkuri* (1995).

(ii) The most consistent results are corresponding to the case 4.

(iii) The results of case 5 show a shift of about 2.5 (m^2 / s^2) with respect to the mean value of w_0 calculated from case1–case4.

(iv) The mean of the geoid potential value w_0 based on GPS observations of Baltic Sea Level project, 2nd campaign, and the orthometric heights of all cases is (62636855.962 $\pm$ 0.536) (m^2 / s^2) and based on the orthometric heights of the case1–case4 is 62636855.515 $\pm$ 0.385 (m^2 / s^2).

Table 4–6: Summary of w_0 values computed from the GPS observations of the Baltic Sea level project, second campaign 1993.4, for the case1–case5.

case	w_0 (m^2 / s^2)	$std(w_0)$ (m^2 / s^2)
case 1	62636854.718	0.3248
case 2	62636855.375	0.4557
case 3	62636855.494	0.4325
case 4	62636856.356	0.3177
case 5	62636857.733	0.3658

5. Baltic Sea Level Project, Third Campaign 1997.4

Here we present the results of w_0 computation based on the GPS observations of the Baltic Sea Level project, 3rd campaign, in five cases introduced before. *Table 5–1–Table 5–5* are presenting gravitational potential $U(\lambda, \phi, u)$, centrifugal potential $V(\lambda, \phi, u)$, free-air gravity potential $\Delta W(\lambda, \phi, u)$ computed based on the orthometric heights of case 1–case 5. A summary of the mean values derived from different cases and the standard deviation of those mean values is given in *Table 5–6*.

Table 5–1: Gravitational potential $U(\lambda, \phi, u)$, centrifugal potential $V(\lambda, \phi, u)$, free-air gravity potential $\Delta W(\lambda, \phi, u; H_0)$, computed based on orthometric heights of the case 1 (*M. Poutanen et al.*, 1999), and gauge value of geoid potential W_0 for the epoch of 1997.4. The W_0 at *Ustka* (Pol) is outlier and removed from the mean.

Station Name	Gravitational Potential $U(\lambda, \phi, u)$ $(m^2 s^{-2})$	Centrifugal Potential $V(\lambda, \phi, u)$ $(m^2 s^{-2})$	Free Air Reduction $\delta W(\lambda, \phi, u)$ $(m^2 s^{-2})$	Gauge Value $(m^2 s^{-2})$
Borkum (Ger)	62598480.049	38330.947	44.888	62636855.884
Degerby (Fin)	62609699.381	27125.102	27.738	62636852.221
Furuögrund (Swe)	62617205.722	19542.016	107.773	62636855.511
Hamina (Fin)	62610584.135	26255.426	16.015	62636855.576
Hanko (Fin)	62609334.840	27467.913	50.254	62636853.007
Helgoland (Ger)	62599593.036	37218.924	44.468	62636856.427
Helsinki (Fin)	62609868.216	26924.835	63.039	62636856.090
Kemi (Fin)	62618330.677	18455.223	69.665	62636855.565
Klagshamn (Swe)	62602014.413	34818.792	20.004	62636853.208
Klaipeda (Lit)	62602168.314	34409.333	276.885	62636854.532
Kronstadt (Rus)	62609616.854	27194.595	–	–
List/Sylt (Ger)	62601102.613	35713.291	40.782	62636856.686
Mäntyluoto (Fin)	62612227.421	24604.328	24.638	62636856.388
Molas (Lit)	62602356.570	34452.784	44.926	62636854.279
Ölands N. U. (Swe)	62605214.649	31601.729	40.514	62636856.892
Raahe (Fin)	62616878.076	19941.164	34.653	62636853.893
Ratan (Swe)	62615930.933	20910.715	15.077	62636856.725
Shepelevo (Rus)	62609584.472	27231.134	–	–
Spikarna (Swe)	62613440.933	23395.355	18.895	62636855.183
Stockholm (Swe)	62608441.995	28295.629	116.888	62636854.512
Swinoujscie (Pol)	62599126.796	37663.497	–	–
Ustka (Pol)	62600356.210	36479.268	14.997	62636850.475
Vaasa (Fin)	62614577.940	22265.875	11.589	62636855.403
Visby (Swe)	62605699.404	31135.656	19.556	62636854.616
Warnemünde (Ger)	62599530.263	37213.313	111.089	62636854.665
			mean $(m^2 s^{-2})$	62636855.108
			std $(m^2 s^{-2})$	0.3239

Table 5–2: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u;H_0)$, computed based on orthometric heights of the case 2 (*J. Kakkuri*, 1995), and gauge value of geoid potential W_0 for the epoch of 1997.4.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598480.049	38330.947	43.229	62636854.226
Degerby (Fin)	62609699.381	27125.102	28.248	62636852.731
Furuögrund (Swe)	62617205.722	19542.016	109.109	62636856.847
Hamina (Fin)	62610584.135	26255.426	16.663	62636856.224
Hanko (Fin)	62609334.840	27467.913	50.794	62636853.547
Helgoland (Ger)	62599593.036	37218.924	–	–
Helsinki (Fin)	62609868.216	26924.835	63.383	62636856.433
Kemi (Fin)	62618330.677	18455.223	70.579	62636856.478
Klagshamn (Swe)	62602014.413	34818.792	20.602	62636853.807
Klaipeda (Lit)	62602168.314	34409.333	–	–
Kronstadt (Rus)	62609616.854	27194.595	–	–
List/Sylt (Ger)	62601102.613	35713.291	38.436	62636854.340
Mäntyluoto (Fin)	62612227.421	24604.328	25.159	62636856.908
Molas (Lit)	62602356.570	34452.784	–	–
Ölands N. U. (Swe)	62605214.649	31601.729	38.453	62636854.830
Raahe (Fin)	62616878.076	19941.164	35.429	62636854.669
Ratan (Swe)	62615930.933	20910.715	16.609	62636858.257
Shepelevo (Rus)	62609584.472	27231.134	42.045	62636857.650
Spikarna (Swe)	62613440.933	23395.355	23.471	62636859.760
Stockholm (Swe)	62608441.995	28295.629	118.086	62636855.710
Swinoujscie (Pol)	62599126.796	37663.497	65.568	62636855.860
Ustka (Pol)	62600356.210	36479.268	–	–
Vaasa (Fin)	62614577.940	22265.875	12.522	62636856.336
Visby (Swe)	62605699.404	31135.656	17.386	62636852.446
Warnemünde (Ger)	62599530.263	37213.313	–	–
			mean (m^2s^{-2})	62636855.635
			std (m^2s^{-2})	0.4350

Table 5–3: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u;H_0)$, computed based on orthometric heights of the case 3 (*J. Kakkuri and M. Poutanen*, 1997), and gauge value of geoid potential W_0 for the epoch of 1997.4. The W_0 at *Ustka* (Pol) is outlier and is not used for the calculation of mean value.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598480.049	38330.947	–	–
Degerby (Fin)	62609699.381	27125.102	29.063	62636853.546
Furuögrund (Swe)	62617205.722	19542.016	108.558	62636856.296
Hamina (Fin)	62610584.135	26255.426	16.496	62636856.057
Hanko (Fin)	62609334.840	27467.913	51.177	62636853.930

Tabelle 5–3: (continued)

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Helgoland (Ger)	62599593.036	37218.924	–	–
Helsinki (Fin)	62609868.216	26924.835	63.471	62636856.522
Kemi (Fin)	62618330.677	18455.223	71.433	62636857.333
Klagshamn (Swe)	62602014.413	34818.792	16.853	62636850.058
Klaipeda (Lit)	62602168.314	34409.333	–	–
Kronstadt (Rus)	62609616.854	27194.595	–	–
List/Sylt (Ger)	62601102.613	35713.291	–	–
Mäntyluoto (Fin)	62612227.421	24604.328	25.296	62636857.046
Molas (Lit)	62602356.570	34452.784	–	–
Ölands N. U. (Swe)	62605214.649	31601.729	40.622	62636857.000
Raahe (Fin)	62616878.076	19941.164	35.832	62636855.072
Ratan (Swe)	62615930.933	20910.715	15.696	62636857.344
Shepelevo (Rus)	62609584.472	27231.134	42.546	62636858.151
Spikarna (Swe)	62613440.933	23395.355	20.182	62636856.470
Stockholm (Swe)	62608441.995	28295.629	118.950	62636856.574
Swinoujscie (Pol)	62599126.796	37663.497	62.025	62636852.317
Ustka (Pol)	62600356.210	36479.268	11.591	62636847.069
Vaasa (Fin)	62614577.940	22265.875	11.972	62636855.786
Visby (Swe)	62605699.404	31135.656	20.027	62636855.087
Warnemünde (Ger)	62599530.263	37213.313	108.831	62636852.407
			mean (m^2s^{-2})	62636855.389
			std (m^2s^{-2})	0.5059

Table 5–4: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u;H_0)$, based on orthometric heights of the case 4 (*M. Poutanen et al.* 1999), and gauge value of geoid potential W_0 for the epoch of 1997.4. The W_0 at *Ustka* (Pol) is outlier and removed from the mean.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598480.049	38330.947	46.242	62636857.238
Degerby (Fin)	62609699.381	27125.102	28.611	62636853.095
Furuögrund (Swe)	62617205.722	19542.016	109.570	62636857.308
Hamina (Fin)	62610584.135	26255.426	17.724	62636857.284
Hanko (Fin)	62609334.840	27467.913	51.795	62636854.548
Helgoland (Ger)	62599593.036	37218.924	45.243	62636857.203
Helsinki (Fin)	62609868.216	26924.835	64.541	62636857.592
Kemi (Fin)	62618330.677	18455.223	72.612	62636858.512
Klagshamn (Swe)	62602014.413	34818.792	21.505	62636854.710
Klaipeda (Lit)	62602168.314	34409.333	278.131	62636855.779
Kronstadt (Rus)	62609616.854	27194.595	46.533	62636857.981
List/Sylt (Ger)	62601102.613	35713.291	41.077	62636856.980
Mäntyluoto (Fin)	62612227.421	24604.328	25.895	62636857.645

Tabelle 5–4: (continued)

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Molas (Lit)	62602356.570	34452.784	45.927	62636855.280
Ölands N. U. (Swe)	62605214.649	31601.729	41.633	62636858.011
Raahe (Fin)	62616878.076	19941.164	37.374	62636856.614
Ratan (Swe)	62615930.933	20910.715	16.982	62636858.631
Shepelevo (Rus)	62609584.472	27231.134	43.086	62636858.691
Spikarna (Swe)	62613440.933	23395.355	20.506	62636856.794
Stockholm (Swe)	62608441.995	28295.629	118.498	62636856.122
Swinoujscie (Pol)	62599126.796	37663.497	65.057	62636855.350
Ustka (Pol)	62600356.210	36479.268	14.869	62636850.347
Vaasa (Fin)	62614577.940	22265.875	13.759	62636857.574
Visby (Swe)	62605699.404	31135.656	20.233	62636855.293
Warnemünde (Ger)	62599530.263	37213.313	111.000	62636854.576
			mean (m^2s^{-2})	62636856.617
			std (m^2s^{-2})	0.2980

Table 5–5: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u;H_0)$, based on orthometric heights of the case 5 (*J. Kakkuri*, 2000), and gauge value of geoid potential W_0 for the epoch of 1997.4. The W_0 at *Ustka* (Pol) is outlier and not included in mean.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\Delta W(\lambda,\phi,u;E$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598480.049	38330.947	47.557	62636858.553
Degerby (Fin)	62609699.381	27125.102	30.192	62636854.675
Furuögrund (Swe)	62617205.722	19542.016	111.407	62636859.145
Hamina (Fin)	62610584.135	26255.426	19.334	62636858.895
Hanko (Fin)	62609334.840	27467.913	53.366	62636856.119
Helgoland (Ger)	62599593.036	37218.924	46.470	62636858.429
Helsinki (Fin)	62609868.216	26924.835	66.132	62636859.183
Kemi (Fin)	62618330.677	18455.223	74.380	62636860.280
Klagshamn (Swe)	62602014.413	34818.792	22.821	62636856.025
Klaipeda (Lit)	62602168.314	34409.333	–	–
Kronstadt (Rus)	62609616.854	27194.595	–	–
List/Sylt (Ger)	62601102.613	35713.291	42.362	62636858.266
Mäntyluoto (Fin)	62612227.421	24604.328	–	–
Molas (Lit)	62602356.570	34452.784	47.252	62636856.605
Ölands N. U. (Swe)	62605214.649	31601.729	43.057	62636859.434
Raahe (Fin)	62616878.076	19941.164	39.201	62636858.441
Ratan (Swe)	62615930.933	20910.715	18.780	62636860.428
Shepelevo (Rus)	62609584.472	27231.134	–	–
Spikarna (Swe)	62613440.933	23395.355	22.214	62636858.503
Stockholm (Swe)	62608441.995	28295.629	120.040	62636857.664

Tabelle 5–5: (continued)

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\Delta W(\lambda,\phi,u;E$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Swinoujscie (Pol)	62599126.796	37663.497	66.274	62636856.567
Ustka (Pol)	62600356.210	36479.268	16.126	62636851.604
Vaasa (Fin)	62614577.940	22265.875	15.704	62636859.518
Visby (Swe)	62605699.404	31135.656	21.676	62636856.736
Warnemünde (Ger)	62599530.263	37213.313	112.227	62636855.803
			mean (m^2s^{-2})	62636857.963
			std (m^2s^{-2})	0.3590

Now let us summarise those results we obtained in five different cases for w_0, based on GPS observation of the Baltic Sea Level project, 3rd campaign, epoch 1997.4, in *Table 5–6*. From a review of *Table 5–6* following conclusions can be made:

(i) The results of w_0 computations based on the data of the third campaign are in general more accurate than the results we obtained for the first and the second campaigns, which indicates the improvement in the accuracy of the GPS observations.

(ii) The most consistent results are corresponding to the heights of case 4.

(iii) The results of case 5 shows a shift of about 2.3 (m^2/s^2) with respect to mean value of w_0 calculated from other four cases.

The weighted mean of the geoid potential value w_0 based on the GPS observations of Baltic Sea Level project, 3rd campaign, and orthometric heights of all cases is (62636856.248 ± 0.525) (m^2/s^2), and based on the orthometric heights of the case1–case4 is 62636855.804 ± 0.378 (m^2/s^2).

Finally, from a review of *Table 3–6*, *Table 4–6*, and *Table 5–6* we can conclude the value $w_0 = (62636855.75 \pm 0.21)(m^2/s^2)$ as our best estimate of the geoid potential value.

Table 5–6: Summary of w_0 values computed from the GPS observations of the Baltic Sea level project, third campaign 1997.4, for case1–case5.

Case	w_0 (m^2/s^2)	$std(w_0)$ (m^2/s^2)
case 1	62636855.108	0.3239
case 2	62636855.635	0.4350
case 3	62636855.389	0.5059
case 4	62636856.617	0.2980
case 5	62636857.963	0.3590

6. Time Derivative of W_0

In the Previous section we presented the value $w_0 = (62636855.75 \quad \pm 0.21)(m^2/s^2)$, from the analysis of w_0 values computed in five cases and three campaigns of GPS observations of Baltic Sea Level project. However, for $\dot{w}_0$ computation we need a different computation strategy. This strategy is motivated by the fact that in relatively short time span from 1990.8 till 1997.4, the only variation in w_0 which can be sensible, is due to eustatic rise. Therefore, to remove the effects of any sources other than variations in sea surface, we computed again the w_0 values for the epochs of 1990.8 and 1993.4 using the orthometric heights of case1–case5 but the GPS coordinates of Baltic Sea Level project, 3rd campaign, epoch 1997.4 only, which are the most accurate GPS observations being made so far at the Baltic Sea.

Based on the new computation, we have come up with the results shown in *Table 6–1–Table 6–5*. From a review of *Table 6–1–Table 6–5* following conclusions can be made:

(i) The most consistent results correspond to case 1, where the heights are computed in their respective national heights without application of regional geoid solutions.

(ii) Within the results of case 1, most consistent results belong to the tide gauge stations of Germany.

(iii) From the $\dot{w}_0$ computations in four stations *Borkum (Ger), Helgoland (Ger), List/Sylt (Ger), Ustka (Pol)* we have computed following estimatation for the time derivative of the geoid potential value "$\dot{w}_0$".

$$\dot{w}_0 = (-0.0099 \pm 0.00079)(m^2/s^2)/year \tag{6.1}$$

Table 6–1: Results of $\dot{w}_0$ computation based on the orthometric heights of case 1. The most consistent results correspond to tide gauge stations *Borkum (Ger), Helgoland (Ger), List/Sylt (Ger), Ustka (Pol).*

Station Name	$\dot{w}_0$ $(m^2s^{-2})/y$ 1993.4-1990.8	$\dot{w}_0$ $(m^2s^{-2})/y$ 1997.4-1993.4	$\dot{w}_0$ $(m^2s^{-2})/y$ 1997.4-1990.8	$\dot{w}_0$ $(m^2s^{-2})/y$ mean
Borkum(Ger)	–0.01153	–0.00975	–0.01045	–0.01058
Degerby(Fin)	–0.05307	2.80830	1.68110	1.47870
Furuögrund(Swe)	0.09076	0.08850	0.08939	0.08955
Hamina(Fin)	0.01884	0.01725	0.01787	0.01799
Hanko(Fin)	0.02653	8.22100	4.99290	4.41350
Helgoland(Ger)	–0.00769	–0.01475	–0.01197	–0.01147
Helsinki(Fin)	0.02269	0.93525	0.57576	0.51123
Kemi(Fin)	0.07576	14.30700	8.70090	7.69460
Klagshamn(Swe)	0	–0.00250	–0.00151	–0.00133
Klaipeda(Lit)	–	0	–	–
Kronstadt(Rus)	–	–	–	–
List/Sylt(Ger)	–0.00384	–0.00975	–0.00742	–0.00700
Mäntyluoto(Fin)	0.06423	0.06150	0.06257	0.06276
Molas(Lit)	–	0	–	–
ÖlandsN.U.(Swe)	0.01538	0.01225	0.01348	0.01370
Raahe(Fin)	–0.76692	3.54100	1.84390	1.53930
Ratan(Swe)	0.09038	0.08600	0.08772	0.08803
Shepelevo(Rus)	–	–	–	–
Spikarna(Swe)	0.07923	0.07600	0.07727	0.07750
Stockholm(Swe)	0.03000	–2.37600	–1.42820	–1.25810
Swinoujscie(Pol)	–0.01115	–	–	–
Ustka(Pol)	–0.01153	–0.00975	–0.01045	–0.01058
Vaasa(Fin)	0.07923	0.07600	0.07727	0.07750
Visby(Swe)	0	0.01475	0.00893	0.00789
Warnemünde(Ger)	–	–24.46700	–	–

Table 6–2: Results of $\dot{w}_0$ computation based on the orthometric heights of case 2. The results from one station to anther are very much different and therefore not acceptable.

Station Name	$\dot{w}_0$ $(m^2s^{-2})/y$ 1993.4-1990.8	$\dot{w}_0$ $(m^2s^{-2})/y$ 1997.4-1993.4	$\dot{w}_0$ $(m^2s^{-2})/y$ 1997.4-1990.8	$\dot{w}_0$ $(m^2s^{-2})/y$ mean
Borkum(Ger)	−1.5477	−0.0685	−0.65121	−0.7558
Degerby(Fin)	0.12846	2.8695	1.7897	1.5959
Furuögrund(Swe)	0.29846	0.0835	0.16818	0.18338
Hamina(Fin)	0.075385	−0.03175	0.010455	0.01803
Hanko(Fin)	0.10962	8.221	5.0256	4.4521
Helgoland(Ger)	–	–	–	–
Helsinki(Fin)	0.19615	0.9475	0.65152	0.59839
Kemi(Fin)	0.20385	14.317	8.7573	7.7594
Klagshamn(Swe)	0.24923	0.09575	0.15621	0.16706
Klaipeda(Lit)	–	–	–	–
Kronstadt(Rus)	–	–	–	–
List/Sylt(Ger)	−0.24923	−0.03425	−0.11894	−0.13414
Mäntyluoto(Fin)	0.048846	0.07125	0.062424	0.06084
Molas(Lit)	–	–	–	–
ÖlandsN.U.(Swe)	0.21154	0.027	0.099697	0.11275
Raahe(Fin)	−1.0958	3.6047	1.753	1.4207
Ratan(Swe)	−0.36654	0.076	−0.098333	−0.12962
Shepelevo(Rus)	–	0.17425	–	–
Spikarna(Swe)	0.17	0.0615	0.10424	0.11191
Stockholm(Swe)	0.47231	−2.4375	−1.2912	−1.0855
Swinoujscie(Pol)	0.31692	10.584	6.5397	5.8137
Ustka(Pol)	–	–	–	–
Vaasa(Fin)	0.21154	0.11775	0.1547	0.16133
Visby(Swe)	0.42269	−0.00975	0.16061	0.19118
Warnemünde(Ger)	–	–	–	–

Table 6–3: Results of $\dot{w}_0$ computation based on the orthometric heights of case 3. The results from one station to anther are very much different and therefore not acceptable.

Station Name	$\dot{w}_0$ $(m^2s^{-2})/y$ 1993.4-1990.8	$\dot{w}_0$ $(m^2s^{-2})/y$ 1997.4-1993.4	$\dot{w}_0$ $(m^2s^{-2})/y$ 1997.4-1990.8	$\dot{w}_0$ $(m^2s^{-2})/y$ mean
Borkum(Ger)	–	–	–	–
Degerby(Fin)	0.12846	2.8695	1.7897	1.5959
Furuögrund(Swe)	0.29846	0.08325	0.16803	0.18325
Hamina(Fin)	0.075769	−0.032	0.010455	0.018075
Hanko(Fin)	0.10962	8.221	5.0256	4.4521
Helgoland(Ger)	–	–	–	–
Helsinki(Fin)	0.19615	0.94775	0.65167	0.59852
Kemi(Fin)	0.20423	14.317	8.7574	7.7596
Klagshamn(Swe)	0.24923	0.09575	0.15621	0.16706
Klaipeda(Lit)	–	–	–	–
Kronstadt(Rus)	–	–	–	–
List/Sylt(Ger)	–	–	–	–

Tabelle 6–3: (continued)

Station Name	$\dot{w}_0$ $(m^2s^{-2})/y$ 1993.4-1990.8	$\dot{w}_0$ $(m^2s^{-2})/y$ 1997.4-1993.4	$\dot{w}_0$ $(m^2s^{-2})/y$ 1997.4-1990.8	$\dot{w}_0$ $(m^2s^{-2})/y$ mean
Mäntyluoto(Fin)	0.049231	0.07125	0.062576	0.061019
Molas(Lit)	–	–	–	–
ÖlandsN.U.(Swe)	0.21154	0.027	0.099697	0.11275
Raahe(Fin)	–1.0954	3.6047	1.7532	1.4208
Ratan(Swe)	–0.36654	0.07625	–0.098182	–0.12949
Shepelevo(Rus)	–	0.17425	–	–
Spikarna(Swe)	0.17	0.0615	0.10424	0.11191
Stockholm(Swe)	0.47231	–2.4375	–1.2912	–1.0855
Swinoujscie(Pol)	0.31692	10.584	6.5397	5.8137
Ustka(Pol)	0.22654	–0.01475	0.080303	0.097364
Vaasa(Fin)	0.21154	0.11775	0.1547	0.16133
Visby(Swe)	0.42269	–0.00975	0.16061	0.19118
Warnemünde(Ger)	71.218	–24.676	13.101	19.881

Table 6–4: Results of $\dot{w}_0$ computation based on the orthometric heights of case 4. The results from one station to anther are very much different and therefore not acceptable.

Station Name	$\dot{w}_0$ $(m^2s^{-2})/y$ 1993.4-1990.8	$\dot{w}_0$ $(m^2s^{-2})/y$ 1997.4-1993.4	$\dot{w}_0$ $(m^2s^{-2})/y$ 1997.4-1990.8	$\dot{w}_0$ $(m^2s^{-2})/y$ mean
Borkum(Ger)	–1.5477	–0.06875	–0.65136	–0.75594
Degerby(Fin)	0.12846	2.8695	1.7897	1.5959
Furuögrund(Swe)	0.29846	0.0835	0.16818	0.18338
Hamina(Fin)	0.075385	–0.032	0.010303	0.017896
Hanko(Fin)	0.10962	8.2208	5.0255	4.4519
Helgoland(Ger)	–0.0038462	0.118	0.07	0.061385
Helsinki(Fin)	0.19654	0.9475	0.65167	0.59857
Kemi(Fin)	0.20385	14.317	8.7574	7.7595
Klagshamn(Swe)	0.24923	0.09575	0.15621	0.16706
Klaipeda(Lit)	–	0.032	–	–
Kronstadt(Rus)	–	–	–	–
List/Sylt(Ger)	–0.24885	–0.0345	–0.11894	–0.1341
Mäntyluoto(Fin)	0.049231	0.07125	0.062576	0.061019
Molas(Lit)	–	0.03675	–	–
ÖlandsN.U.(Swe)	0.21154	0.027	0.099697	0.11275
Raahe(Fin)	–1.0954	3.6047	1.7532	1.4208
Ratan(Swe)	–0.36654	0.07625	–0.098182	–0.12949
Shepelevo(Rus)	–	0.17425	–	–
Spikarna(Swe)	0.16962	0.0615	0.10409	0.11174
Stockholm(Swe)	0.47192	–2.4375	–1.2914	–1.0856
Swinoujscie(Pol)	0.31731	10.584	6.5398	5.8139
Ustka(Pol)	0.22654	–0.01475	0.080303	0.097364
Vaasa(Fin)	0.21154	0.118	0.15485	0.16146
Visby(Swe)	0.42269	–0.00975	0.16061	0.19118
Warnemünde(Ger)	71.218	–24.676	13.101	19.881

Table 6–5: Results of $\dot{w}_0$ computation based on the orthometric heights of case 5. The results from one station to anther are very much different and therefore not acceptable.

Station Name	$\dot{w}_0$ $(m^2s^{-2})/y$ 1993.4-1990.8	$\dot{w}_0$ $(m^2s^{-2})/y$ 1997.4-1993.4	$\dot{w}_0$ $(m^2s^{-2})/y$ 1997.4-1990.8	$\dot{w}_0$ $(m^2s^{-2})/y$ mean
Borkum(Ger)	–1.5477	–0.06875	–0.65136	–0.75594
Degerby(Fin)	0.12808	2.8695	1.7895	1.5957
Furuögrund(Swe)	0.29846	0.0835	0.16818	0.18338
Hamina(Fin)	0.075385	–0.03175	0.010455	0.01803
Hanko(Fin)	0.10962	8.2208	5.0255	4.4519
Helgoland(Ger)	–0.0038462	0.11775	0.069848	0.061251
Helsinki(Fin)	0.19615	0.94775	0.65167	0.59852
Kemi(Fin)	0.20423	14.317	8.7574	7.7596
Klagshamn(Swe)	0.24923	0.0955	0.15606	0.16693
Klaipeda(Lit)	–	–	–	–
Kronstadt(Rus)	–	–	–	–
List/Sylt(Ger)	–0.24923	–0.03425	–0.11894	–0.13414
Mäntyluoto(Fin)	–	–	–	–
Molas(Lit)	–	0.03675	–	–
ÖlandsN.U.(Swe)	0.21115	0.027	0.099545	0.11257
Raahe(Fin)	–1.0954	3.6047	1.7532	1.4208
Ratan(Swe)	–0.36615	0.076	–0.098182	–0.12945
Shepelevo(Rus)	–	–	–	–
Spikarna(Swe)	0.17	0.0615	0.10424	0.11191
Stockholm(Swe)	0.47231	–2.4375	–1.2912	–1.0855
Swinoujscie(Pol)	0.31731	10.584	6.5398	5.8139
Ustka(Pol)	0.22654	–0.01475	0.080303	0.097364
Vaasa(Fin)	0.21154	0.11775	0.1547	0.16133
Visby(Swe)	0.42269	–0.00975	0.16061	0.19118
Warnemünde(Ger)	71.218	–24.676	13.101	19.881

7. Height Datum Difference of the Countries Around the Baltic Sea

Indeed, the results we have presented in *Table 5–1* provide us with a means to estimate the difference between the height datums of countries around the Baltic Sea. If now we rearrange *Table 5–1* in a way that the w_0 values of the each country be kept together, we arrive at *Table 7–1*, which helps us to see the variation of w_0 values from one country to the other. As it is shown in *Table 7–1* for each country we have computed a reference potential by averaging the two closest w_0 at tide gauge stations of that country. Based on those reference potential values we computed the height datum difference between Finland, Germany, Lithuania and Sweden as shown in *Table 7–2* in gravity space in terms of potential difference, and in *Table 7–3* in geometry space, in terms of height difference. To convert the potential differences into height differences (i.e. to transformation from gravity space to geometry space) we use the mean value of vertical gradient of gravity –9.81802523 m/s^2 in Baltic Sea area, based on ellipsoidal harmonic model of degree/order 360/360. We leave the details of computation of vertical gradient of gravity to the next section.

Table 7–1: Gravitational potential $U(\lambda,\phi,u)$, centrifugal potential $V(\lambda,\phi,u)$, free-air gravity potential $\Delta W(\lambda,\phi,u;H_0)$, computed based on orthometric heights of the case 1 (*M. Poutanen et al.*, 1999), and gauge value of geoid potential W_0 for the epoch of 1997.4. The highlighted stations are those used for computing the reference potentials.

Station Name	Gravitational Potential $U(\lambda,\phi,u)$ (m^2s^{-2})	Centrifugal Potential $V(\lambda,\phi,u)$ (m^2s^{-2})	Free Air Reduction $\delta W(\lambda,\phi,u)$ (m^2s^{-2})	Gauge Value W_0 (m^2s^{-2})
Borkum (Ger)	62598480.049	38330.947	44.888	62636855.884
Helgoland (Ger)	62599593.036	37218.924	44.468	62636856.427
List/Sylt (Ger)	62601102.613	35713.291	40.782	62636856.686
Warnemünde (Ger)	62599530.263	37213.313	111.089	62636854.665
			mean (m^2s^{-2})	62636856.557
			std (m^2s^{-2})	0.129
Degerby (Fin)	62609699.381	27125.102	27.738	62636852.221
Hamina (Fin)	62610584.135	26255.426	16.015	62636855.576
Hanko (Fin)	62609334.840	27467.913	50.254	62636853.007
Helsinki (Fin)	62609868.216	26924.835	63.039	62636856.090
Kemi (Fin)	62618330.677	18455.223	69.665	62636855.565
Mäntyluoto (Fin)	62612227.421	24604.328	24.638	62636856.388
Raahe (Fin)	62616878.076	19941.164	34.653	62636853.893
Vaasa (Fin)	62614577.940	22265.875	11.589	62636855.403
			mean (m^2s^{-2})	62636855.515
			std (m^2s^{-2})	0.056
Furuögrund (Swe)	62617205.722	19542.016	107.773	62636855.511
Klagshamn (Swe)	62602014.413	34818.792	20.004	62636853.208
Ölands N. U. (Swe)	62605214.649	31601.729	40.514	62636856.892
Ratan (Swe)	62615930.933	20910.715	15.077	62636856.725
Spikarna (Swe)	62613440.933	23395.355	18.895	62636855.183
Stockholm (Swe)	62608441.995	28295.629	116.888	62636854.512
Visby (Swe)	62605699.404	31135.656	19.556	62636854.616
			mean (m^2s^{-2})	62636856.809
			std (m^2s^{-2})	0.083
Klaipeda (Lit)	62602168.314	34409.333	276.885	62636854.532
Molas (Lit)	62602356.570	34452.784	44.926	62636854.279
			mean (m^2s^{-2})	62636854.406
			std (m^2s^{-2})	0.126
Kronstadt (Rus)	62609616.854	27194.595	–	–
Shepelevo (Rus)	62609584.472	27231.134	–	–
Swinoujscie (Pol)	62599126.796	37663.497	–	–
Ustka (Pol)	62600356.210	36479.268	14.997	62636850.475

Table 7–2: Datum difference between the countries around Baltic Sea, in potential units.

	Finland	Germany	Lithuania	Sweden
Finland	0	(–1.042 ± 0.070) (m^2s^{-2})	(1.109 ± 0.069) (m^2s^{-2})	(–1.294 ± 0.050) (m^2s^{-2})
Germany	(1.042 ± 0.070) (m^2s^{-2})	0	(2.151 ± 0.090) (m^2s^{-2})	(–0.252 ± 0.077) (m^2s^{-2})
Lithuania	(–1.109 ± 0.069) (m^2s^{-2})	(–2.151 ± 0.090) (m^2s^{-2})	0	(–2.403 ± 0.075) (m^2s^{-2})
Sweden	(1.294 ± 0.050) (m^2s^{-2})	(0.252 ± 0.077) (m^2s^{-2})	(2.403 ± 0.075) (m^2s^{-2})	0

Table 7–3: Datum difference between the countries around Baltic Sea, in length unit, based on mean value of vertical gradient of gravity –9.81802523 m/s^2 in Baltic Sea area.

	Finland	Germany	Lithuania	Sweden
Finland	0	(–0.106 ± 0.007) (m)	(0.113 ± 0.007) (m)	(–0.132 ± 0.005) (m)
Germany	(0.106 ± 0.007) (m)	0	(0.219 ± 0.009) (m)	(–0.026 ± 0.008) (m)
Lithuania	(–0.113 ± 0.007) (m)	(–0.219 ± 0.009) (m)	0	(–0.245 ± 0.008) (m)
Sweden	(0.132 ± 0.005) (m)	(0.026 ± 0.008) (m)	(0.245 ± 0.008) (m)	0

8. Sea Surface Topography Map of Baltic Sea

Having accessed to the MSL information at various tide gauge stations around Baltic Sea we have opportunity to derive the *Sea Surface Topography (SST)* of the Baltic Sea. Here we use the GPS observations of the Baltic Sea Level project, 3rd campaign, which are the most accurate ones, and the orthometric heights of the case 1, which are the directly observed heights above MSL in height datum of various countries (see *Table 5–1*). Of course now that we have estimated the datum difference between the countries around Baltic Sea we can unify the datum of the orthometric heights of the case 1.

The difference between the w_0 values presented in *Table 5–1* and the average value $w_0 =$ $(62636855.75 \ \ \pm 0.21)(m^2/s^2)$, provide us with the apparent SST, i.e., the deviation of the sea surface from the geoid in various tide gauge stations around the Baltic Sea, plus the height datum difference. These potential deviations can be converted into metric units according to the transformation relation outlined in *Table 8–1.*

Table 8–1: Transformation relation of potential difference into height difference.

"decomposition of actual geoid potential value w_0 into the apparent geoid potential value at the tide gauges W_{0i}, and the disturbing part δW"

$$\begin{aligned} w_0 &= W_{0i} + \delta W \\ &= W_{0i} + \nabla_N W \cdot (u_0 - u) \\ &= W_{0i} + \frac{1}{\sqrt{g_{uu}}} \frac{\partial W}{\partial u} \cdot \sqrt{g_{uu}} (u_0 - u) \\ &= W_{0i} + \frac{1}{\sqrt{g_{uu}}} \frac{\partial W}{\partial u} \cdot \Delta u^{(1)} \end{aligned} \tag{8.1}$$

"the physical height difference

(i.e. SST)"

$$\Delta u^{(1)} = \frac{(w_0 - W_{0i}) \sqrt{g_{uu}}}{\dfrac{\partial W}{\partial u}} \tag{8.2}$$

"subject to"

$$\begin{aligned} \frac{\partial W}{\partial u} = \sum_{n=0}^{360} \sum_{m=-n}^{+n} u_{nm} \frac{1}{Q^*_{n|m|}(\frac{b}{\varepsilon})} \frac{\partial Q^*_{n|m|}(\frac{u}{\varepsilon})}{\partial u} e_{nm}(\lambda, \phi) \\ + \omega^2 u \sqrt{u^2 + \varepsilon^2} \cos^2 \phi \end{aligned} \tag{8.3}$$

As shown in *Table 8–1* the geoid potential value w_0 can be decomposed into the computed (or apparent) geoid potential value at the various tide gauges W_{0i} (index i runs from one to the total number of tide gauge stations), and the disturbing part δW. The disturbing part is due to the *Sea Surface Topography* $(u_0 - u)$. The Sea Surface Topography $(u_0 - u)$ is in terms of Jacobi ellipsoidal coordinates, and as such is not a physical component. However, $\Delta u^{(1)} = \sqrt{g_{uu}}(u_0 - u)$, derived by using the directional derivative operator (see(8.1) – (8.2)), is the required physical height. For the derivative of the potential with respect to u appearing in (8.2), we use the high-resolution model of (8.3).

Table 8–2 presents the computed SST, $\Delta u^{(1)}$, of the Baltic Sea at various tide gauge stations. We have also corrected the SST of tide gauge stations for the difference between the national height datums, to reach to the last column of *Table* 8–2, *which* shows the SST of the Baltic Sea Level tide gauges in the German height datum. Note that the SST could be more stable with respect to the German height datum than the height datums of the countries at the northern part of the Baltic Sea, for example, especially due to the runoff of rivers in the north in the spring period. That is why we have computed the SST of the Baltic Sea in the height datum of Germany. A contour map plot of computed SST of the Baltic Sea is shown in *Figure 8–1.*

Table 8–3 shows the comparison of the computed SST for Baltic Sea as explained above and the one computed by *J. Kakkuri M. Poutanen* (1997). The difference between the two SST solutions is $(0.011 \pm 0.054)\,m$.

Table 8–2: Vertical gradient of Potential, difference between apparent W_{0i} based on height of the stations in their respective national height systems and average value of geoid potential w_0, apparent SST at the tide gauge stations of Baltic Sea level project in different height datums, corrections to German's height datum, corrected SST in German height datum.

Station Name	$\frac{1}{\sqrt{g_{uu}}}\frac{\partial W}{\partial u}$ (m/s^2)	$(w_0 - W_0)$ (m^2/s^2)	$\Delta u^{(1)} = \frac{(w_0 - W_0)}{\frac{1}{\sqrt{g_{uu}}}\frac{\partial W}{\partial u}}$ (m)	Correction to German height datum (m)	Corrected SST (m)
Borkum(Ger)	–9.81363626	–0.134	0.0137	0	–0.014
Degerby(Fin)	–9.81862941	3.529	–0.3594	0.106	0.253
Furuögrund(Swe)	–9.82251837	0.239	–0.0243	–0.026	0.050
Hamina(Fin)	–9.81923277	0.174	–0.0177	0.106	–0.088
Hanko(Fin)	–9.81899359	2.743	–0.2794	0.106	0.173
Helgoland(Ger)	–9,81407564	–0.677	0.0690	0	–0.069
Helsinki(Fin)	–9.81916005	–0.340	0.0346	0.106	–0.141
Kemi(Fin)	–9.82306382	0.185	–0.0188	0.106	–0.087
Klagshamn(Swe)	–9.81537974	2.542	–0.2590	–0.026	0.285
Klaipeda(Lit)	–9.81547717	1.218	–0.1241	0.219	–0.095
Kronstadt(Rus)	–9.81913536	–	–	–	–
List/Sylt(Ger)	–9.81521803	–0.936	0.0954	0	–0.095
Mäntyluoto(Fin)	–9.82002342	–0.638	0.0650	0.106	–0.171
Molas(Lit)	–9.81549953	1.471	–0.1499	0.219	–0.069
ÖlandsN.U.(Swe)	–9.81683093	-1.142	0.1163	–0.026	–0.090
Raahe(Fin)	–9.82228525	1.857	–0.1891	0.106	0.083
Ratan(Swe)	–9.82207194	–0.975	0.0993	–0.026	–0.073
Shepelevo(Rus)	–9.81897667	–	–	–	–
Spikarna(Swe)	–9.82068805	0.567	–0.0577	–0.026	0.084
Stockholm(Swe)	–9.81840031	1.238	–0.1261	–0.026	0.152
Swinoujscie(Pol)	–9.81406739	–	–	–	–
Ustka(Pol)	–9.81472950	5.275	–0.5375	–	–
Vaasa(Fin)	–9.82096810	0.347	–0.0353	0.106	–0.071
Visby(Swe)	–9.81722771	1.134	–0.1155	–0.026	0.142
Warnemünde(Ger)	–9.81434182	1.085	–0.1106	0	0.111

Table 8–3: Comparison of our SST with the SST computed by *J. Kakkuri M. Poutanen* (1997) for the Baltic Sea.

Station Name	our SST (m)	SST (*J. Kakkuri M. Poutanen*, 1997)	Difference between two SST's
Borkum(Ger)	–0.014	–	–
Degerby(Fin)	0.253	0.095	0.158
Furuögrund(Swe)	0.050	0.054	–0.004
Hamina(Fin)	–0.088	0.052	–0.140
Hanko(Fin)	0.173	0.079	0.094
Helgoland(Ger)	–0.069	–	–
Helsinki(Fin)	–0.141	0.024	–0.165
Kemi(Fin)	–0.087	0.146	–0.233
Klagshamn(Swe)	0.285	–0.363	0.648
Klaipeda(Lit)	–0.095	–	–

Tabelle 8–3: (continued)

Station Name	our SST (m)	SST (*J. Kakkuri M. Poutanen,* 1997)	Difference between two SST's
Kronstadt(Rus)	–	–	–
List/Sylt(Ger)	–0.095	–	–
Mäntyluoto(Fin)	–0.171	0.044	–0.215
Molas(Lit)	–0.069	–	–
ÖlandsN.U.(Swe)	–0.090	–0.03	–0.060
Raahe(Fin)	0.083	0.067	0.016
Ratan(Swe)	–0.073	0.041	–0.114
Shepelevo(Rus)	–	0.260	–
Spikarna(Swe)	0.084	0.116	–0.032
Stockholm(Swe)	0.152	0.199	–0.047
Swinoujscie(Pol)	–	–0.300	–
Ustka(Pol)	–	–0.344	–
Vaasa(Fin)	–0.071	–0.01	–0.061
Visby(Swe)	0.142	0.050	0.092
Warnemünde(Ger)	0.111	–0.143	0.254

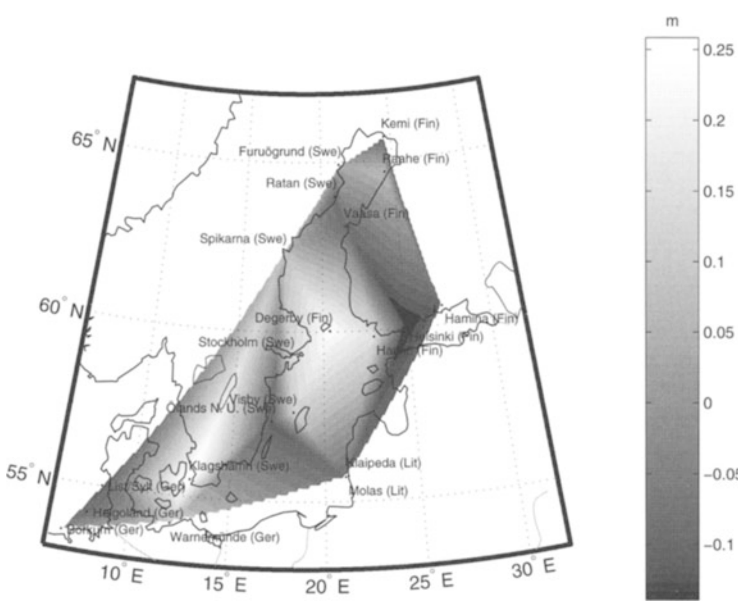

Figure 8–1: The computed SST map of Baltic Sea in German Height Datum. Equidistant *Conic Map Projection* with two standard parallels: $55^\circ N$ and $65^\circ N$, and reference ellipsoid of *WGD2000*.

9. Conclusions

Let us summaries the results we obtained in various sections as follows.

(i) Our best estimate for w_0 value based on the GPS observations of the Baltic Sea Level project, 1st, 2nd, and 3rd campaigns, and tide gauge observations is

$$w_0 = (62636855.75 \pm 0.21)(m^2 / s^2)$$

(ii) Our best estimate for $\dot{w}_0$ value based on the GPS observations of the Baltic Sea Level project, 1st, 2nd, and 3rd campaigns, and tide gauge observations is

$$\dot{w}_0 = (-0.0099 \pm 0.00079)(m^2 / s^2) / year$$

or

$$\dot{w}_0 / \overline{\gamma} = 1.0(mm / year)$$

(iii) Amongst the different geoid solutions proposed for Baltic Sea the one introduced in *M. Poutanen et al.* (1999) is the most accurate one.

(iv) The geoid proposed by *J. Kakkuri* (2000) has a shift of approximately 0.244 m.

(v) As the comparison with already published w_0 values, we refer *Table 9–1.*

Finally, based on the results obtain we can conclude that our method is quite successful for the computation of fundamental *geodetic parameter* w_0, the potential value of *Gauss-Listing* geoid, as well as its *time derivative* $\dot{w}_0$. Besides, the proposed methodology can also be quite helpful in *(i)* unification of the *National Height Datum*s *(ii)* computation of the high-resolution *Sea Surface Topography maps*, and *(iii)* accuracy estimation of the geoid solutions tailored to Sea areas.

Table 9–1: Comparison of potential value of geoid w_0 computed by various authors.

Author	w_0 value (m^2/s^2)	Data source / Computation method
D. Nesvorný, and Z. Šíma (1994)	62 636 857.5 ± 1.0	satellite altimetry data
Burša et al. (1997a)	62 636 855.72 ± 0.5	satellite altimetry data, gauge stations values
Burša et al. (1997b)	62 636 855.80 ± 0.5	satellite altimetry data, gauge stations values
Grafarend and Ardalan (1997)	62 636 855.8 ± 3.6	ellipsoidal harmonic expansion and, and tide gauge information of Baltic Sea, GPS observations of BSL projects 2nd campaign
Burša et al. (1998)	62 636 855.611 ± 0.5	TOPEX/POSEIDON altimeter data
Burša et al. (2000)	62 636 856.0 ± 0.5	TOPEX/POSEIDON altimeter data
Our results	62 636 855.75 ± 0.21	ellipsoidal harmonic expansion, and tide gauge information of Baltic Sea, GPS observations of BSL projects 1st 2nd and 3rd campaigns

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Appendix A: Spheroidal Coordinates

It has been revealed by the Great Early 18th Century Expeditions that the Earth is *not* geometrically a sphere, but nearly an oblate ellipsoid-of-revolution $\mathbb{E}^2_{a,a,b}$. (See for example *J. Kakkuri et al* (1986), *J.R. Smith* (1986) and *E. Tobé* (1986) as the historical review of the progress in determination of the shape of the *Earth*). Therefore, representation of gravity field of the Earth in terms of ellipsoidal harmonics is more accurate and even more convenient than for example spherical coordinates and spherical harmonics.

Here briefly we will review the main features of ellipsoidal coordinates and ellipsoidal harmonic expansion and invite the interested readers for more details to a visit to *N. Thong and W. Grafarend* (1989).

Definition A–1: (spheroidal coordinates $\{\lambda,\phi,u\}$)

In terms of *ellipsoidal coordinates* $\{\lambda,\phi,u\}$, a point in space can be located as the intersection point of the following family of surfaces.

(i) the family of confocal, oblate spheroids

$$\mathbb{E}^2_{\sqrt{u^2+\varepsilon^2},u} := \left\{\mathbf{x}\in\mathbb{R}^3 \mid \frac{x^2+y^2}{u^2+\varepsilon^2}+\frac{z^2}{u^2}=1,\ u\in(0,+\infty),\ \varepsilon^2:=a^2-b^2\right\} \tag{A.1}$$

(ii) the family of confocal half hyperboloids

$$\mathbb{H}^2_{\varepsilon\cos\phi,\varepsilon\sin\phi} := \left\{\mathbf{x}\in\mathbb{R}^3 \mid \frac{x^2+y^2}{\varepsilon^2\cos^2\phi}-\frac{z^2}{\varepsilon^2\sin^2\phi}=1,\ \phi\in[-\frac{\pi}{2},\frac{\pi}{2}],\ \phi\neq 0\right\} \tag{A.2}$$

(iii) the family of half planes

$$\mathbb{P}^2_{\cos\lambda,\sin\lambda} := \left\{\mathbf{x}\in\mathbb{R}^3 \mid y = x\tan\lambda,\ \lambda\in[0,2\pi]\right\} \tag{A.3}$$

As shown in *Figure A–1* longitude λ gives *orientation* to the half planes, latitude ϕ is the *inclination* of asymptotes of the confocal half hyperboloids, and the elliptic coordinate *u* is the *semi-minor axis* of confocal oblate spheroids (confocal, oblate ellipsoids of revolution).

Table A–1: Forward and backward transformation ofCartesian coordinates $\{x, y, z\}$ into ellipsoidal coordinates $\{\lambda, \phi, u\}$

(i) Forward transformation

$$\{\lambda, \phi, u\} \mapsto \{x, y, z\}$$
$$\begin{aligned} x &= \sqrt{u^2 + \varepsilon^2} \cos\phi \cos\lambda \\ y &= \sqrt{u^2 + \varepsilon^2} \cos\phi \sin\lambda \\ z &= u \sin\phi \end{aligned} \tag{A.4}$$

(ii) Backward transformation of $\{x, y, z\} \mapsto \{\lambda, \phi, u\}$

$$\lambda = \begin{cases} \arctan\frac{y}{x} \text{ for } x > 0 \text{ and } y \geq 0 \\ \arctan\frac{y}{x} + \pi & \text{for } x < 0 \text{ and } y \neq 0 \\ \arctan\frac{y}{x} + 2\pi & \text{for } x > 0 \text{ and } y < 0 \\ \frac{\pi}{2} & \text{for } x = 0 \text{ and } y > 0 \\ 3\frac{\pi}{2} & \text{for } x = 0 \text{ and } y < 0 \end{cases} \tag{A.5}$$

$$\begin{aligned} \phi = (\operatorname{sgn} z) \arcsin\Big\{ \frac{1}{2\varepsilon^2} [\varepsilon^2 - (x^2 + y^2 + z^2) \\ + \sqrt{(x^2 + y^2 + z^2 - \varepsilon^2)^2 + 4\varepsilon^2 z^2}] \Big\}^{1/2} \end{aligned} \tag{A.6}$$

$$\begin{aligned} u = \{\frac{1}{2}[x^2 + y^2 + z^2 - \varepsilon^2 \\ + \sqrt{(x^2 + y^2 + z^2 - \varepsilon^2)^2 + 4\varepsilon^2 z^2}]\}^{1/2} \end{aligned} \tag{A.7}$$

Definition A–2: Basic geometry of *ellipsoidal coordinates* $\{\lambda, \phi, u\}$

(i) *Jacobi matrix of the forward transformation* $\{\lambda, \phi, u\} \mapsto \{x, y, z\}$

From equation (A.4) Jacobi matrix " J " of the transformation from ellipsoidal coordinates $\{\lambda, \phi, u\}$ into Cartesian coordinates $\{x, y, z\}$ can be obtained as follows.

$$J := \begin{bmatrix} X_\lambda & X_\phi & X_u \\ Y_\lambda & Y_\phi & Y_u \\ Z_\lambda & Z_\phi & Z_u \end{bmatrix} \tag{A.8}$$

The partial derivatives used in (A.8) are defined as

$$\begin{aligned} X_\lambda &= D_\lambda X = -\sqrt{u^2 + \varepsilon^2} \cos\phi \sin\lambda \\ Y_\lambda &= D_\lambda Y = \sqrt{u^2 + \varepsilon^2} \cos\phi \cos\lambda \\ Z_\lambda &= D_\lambda Z = 0 \\ X_\phi &= D_\phi X = -\sqrt{u^2 + \varepsilon^2} \sin\phi \cos\lambda \\ Y_\phi &= D_\phi Y = -\sqrt{u^2 + \varepsilon^2} \sin\phi \sin\lambda \\ Z_\phi &= D_\phi Z = u \cos\phi \end{aligned}$$

$$
\begin{aligned}
X_u &= D_u X = \frac{u}{\sqrt{u^2+\varepsilon^2}} \cos\phi \cos\lambda \\
Y_u &= D_u Y = \frac{u}{\sqrt{u^2+\varepsilon^2}} \cos\phi \sin\lambda \\
Z_u &= D_u Z = \sin\phi\,.
\end{aligned}
$$

(ii) *The metric tensor*

$$
dS^2 = [d\lambda, d\phi, du] J^* J \begin{bmatrix} d\lambda \\ d\phi \\ du \end{bmatrix} \tag{A.9}
$$

$$
G := J^* J \begin{bmatrix} (u^2+\varepsilon^2)\cos^2\phi & 0 & 0 \\ 0 & u^2+\varepsilon^2\sin^2\phi & 0 \\ 0 & 0 & (u^2+\varepsilon^2\sin^2\phi)/(u^2+\varepsilon^2) \end{bmatrix} := g_{nm}\ \forall n,m = 1,2,3 \tag{A.10}
$$

(iii) *Laplacian*

$$
\begin{aligned}
\Delta &= \frac{1}{\sqrt{g}}\left\{\frac{\partial}{\partial\lambda}\left(\frac{\sqrt{g}}{g_{11}}\frac{\partial}{\partial\lambda}\right) + \frac{\partial}{\partial\phi}\left(\frac{\sqrt{g}}{g_{22}}\frac{\partial}{\partial\phi}\right) + \frac{\partial}{\partial u}\left(\frac{\sqrt{g}}{g_{33}}\frac{\partial}{\partial u}\right)\right\} \\
&= \frac{1}{u^2+\varepsilon^2\sin^2\phi}\left\{\frac{u^2+\varepsilon^2\sin^2\phi}{(u^2+\varepsilon^2)\cos^2\phi}\frac{\partial^2}{\partial\lambda^2} - \tan\phi\frac{\partial}{\partial\phi}\right. \\
&\quad \left. + \frac{\partial^2}{\partial\phi^2} + 2u\frac{\partial}{\partial u} + (u^2+\varepsilon^2)\frac{\partial^2}{\partial u^2}\right\}
\end{aligned} \tag{A.11}
$$

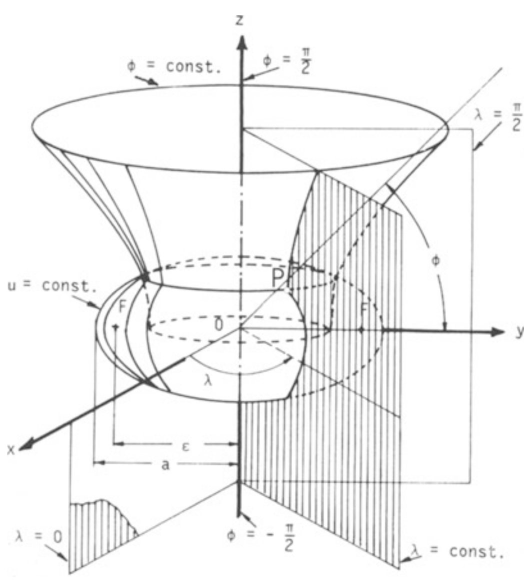

Figure A–1: Spheroidal Coordinates $\{\lambda, \phi, u\}$. The coordinate surfaces are of the type: *(i)* spheroids (u = const.), *(ii)* half hyperboloids of one sheet (ϕ = cost.) and, *(iii)* half planes (λ = const.)

Appendix B: Normalised Associated Legendre Functions of First and Second Kind

Here we define the normalised associated Legendre functions of the first kind $P^*_{nm}(\sin\phi)$ by means of recurrence relations as follows.

$$P^*_{nn}(\sin\phi) = \frac{\sqrt{2n+1}}{\sqrt{2n}}\cos\phi\, P^*_{n-1,n-1}(\sin\phi) \tag{B.1}$$

$$P^*_{n,n-1}(\sin\phi) = \frac{\sqrt{2n+1}}{\sqrt{2(n-1)}}\cos\phi\, P^*_{n-1,n-2}(\sin\phi) \tag{B.2}$$

$$P^*_{nm}(\sin\phi) = \frac{\sqrt{4n^2-1}}{\sqrt{n^2-m^2}}\sin\phi\, P^*_{n-1,m}(\sin\phi) - \frac{\sqrt{(2n+1)(n+m-1)(n-m-1)}}{\sqrt{(n^2-m^2)(2n-3)}} P^*_{n-2,m}(\sin\phi) \tag{B.3}$$

subject to

$$\forall n \in [3,\infty) \text{ and } m \in [0, n-2]$$

with starting values

$$P^*_{00}(\sin\phi) = 1 \tag{B.4}$$

$$P^*_{10}(\sin\phi) = \sqrt{3}\sin\phi \tag{B.5}$$

$$P^*_{11}(\sin\phi) = \sqrt{3}\cos\phi \tag{B.6}$$

$$P^*_{20}(\sin\phi) = \frac{\sqrt{5}}{2}(3\sin^2\phi - 1) \tag{B.7}$$

$$P^*_{21}(\sin\phi) = \sqrt{15}\sin\phi\cos\phi \tag{B.8}$$

$$P^*_{22}(\sin\phi) = \frac{\sqrt{15}}{2}\cos^2\phi \tag{B.9}$$

The associated Legendre functions of the second kind can be defined by an integral relation of the type

$$Q^*_{nm}(\frac{u}{\varepsilon}) = i^{n+1} Q_{nm}(i\frac{u}{\varepsilon}) \tag{B.10}$$

$$Q_{nm}(i\frac{u}{\varepsilon}) = \frac{(-1)^m 2^n (n+m)!\,m!}{i^{n+1}(n-m)!(2m)!}(\frac{u^2+\varepsilon^2}{\varepsilon^2})^{m/2} \cdot \int_0^\infty \frac{\sinh^{2m}\tau\, d\tau}{(\frac{u}{\varepsilon} + \frac{\sqrt{u^2+\varepsilon^2}}{\varepsilon}\cosh\tau)^{n+m+1}} \tag{B.11}$$

However, in practice instead of the above integral formulas the associated Legendre functions of the second kind are calculated via the recursive relations which enjoy the numerical stable, especially for the higher degrees and orders (*N. Thong and E. Grafarend*, 1989, *G. Sona*, 1996)

$$Q^*_{n|m|}(\frac{u}{\varepsilon}) = \sum_{k=0}^{k_{\max}} Q^*_{n|m|k}(u) \tag{B.12}$$

$$Q^*_{n|m|k}(u) = \frac{\varepsilon^2(1-n-\mid m\mid -2k)(n+\mid m\mid +2k)}{2k(2n+2k+1)u^2} Q^*_{n|m|k-1}(u) \quad \forall\, k \geq 1 \tag{B.13}$$

$$Q^*_{n|m|0}(u) = \cosh^{|m|}\eta(\frac{a}{u})^{n+1} \qquad \forall\ n\in\mathbb{N},\ m\in[-n,n]\subset\mathbb{Z} \tag{B.14}$$

The summation (B.12) is continued until

$$Q^*_{n|m|k_{\max}}(u) - Q^*_{n|m|k_{\max-1}}(u) < \sigma \tag{B.15}$$

with starting values for $n = 0, 1, 2$ and $m = 0$

$$Q^*_0(\frac{u}{\varepsilon}) = \operatorname{arc}\cot(\frac{u}{\varepsilon}) \tag{B.16}$$

$$Q^*_1(\frac{u}{\varepsilon}) = 1 - \frac{u}{\varepsilon}\operatorname{arc}\cot(\frac{u}{\varepsilon}) \tag{B.17}$$

$$Q^*_2(\frac{u}{\varepsilon}) = \frac{1}{2}[(3\frac{u^2}{\varepsilon^2}+1)\operatorname{arc}\cot(\frac{u}{\varepsilon}) - 3\frac{u}{\varepsilon}] \tag{B.18}$$

In (B.15) σ can be selected according to required accuracy. In our calculations double precision accuracy, i.e., σ =1E–16 was adopted.

Appendix C: Transformation of Spherical Harmonic Coefficients into Spheroidal/Ellipsoidal Harmonic Coefficients

Nowadays it is a common practice to represent the "*Standard Gravity Earth Models*" in terms of spherical harmonics. Fortunately, precise transformation relations between spherical and ellipsoidal harmonic coefficients are available and therefore one can transfer the spherical harmonic coefficients into ellipsoidal ones without any loss of accuracy. *Lemma C–1* offers a summary of the transformation of spherical harmonic coefficients into ellipsoidal harmonic coefficients according to *C. Jekeli* (1981, 1988). In conjunction with the ellipsoidal harmonics, contribution made by *D. Gleason* (1988, 1989), *G. Sona* (1996) and *J. Yu and H. Cao* (1996) can also be acknowledged.

Lemma C–1: (Transformation of spherical harmonic coefficients into ellipsoidal harmonic coefficients)

Spherical harmonic coefficients, $u_{n,\,m}(spherial)$, can be uniquely transformed into ellipsoidal harmonic coefficients, $u_{n,\,m}(ellipsoid)$ $Q^*_{n,\,|m|}(\frac{b}{\varepsilon})$ are *associated Legendre functions of the second kind*, see equation (B.12) for the recursive relation.

As one can read from (C.1)–(C.5) any ellipsoidal harmonic coefficient, $u_{n,\,m}(spheroid)$ is equal to the spherical harmonic coefficient of the same degree and order $u_{n,\,m}(sphere)$ plus a linear combination of spherical harmonic coefficients of the lower degree but the same order.

For details on transformation of spherical harmonic coefficients into ellipsoidal ones, we refer to *A. Ardalan and E. Grafarend* (2000). The ellipsoidal harmonic coefficients in different permanent tide systems can be accessed also from http://www.uni-stuttgart.de/gi/research/index.html# Projects.

$$u_{n,\,m}(ellipsoidal) = Q^*_{n,\,|m|}(\frac{b}{\varepsilon})\sum_{l=0}^{(n-m)/2}\lambda_{n,\,|m|,\,l}\,u_{n-2|m|,\,|m|}(spherical) \tag{C.1}$$

$$\lambda_{n,m,l} = \frac{(2n-2l)!\,n!}{(2n)!\,l!\,(n-1)!}\left[\frac{(2n-4l+1)(n-m)(n+m)!}{(2n+1)(n-2l+m)!(n-2l-m)!}\right]^{1/2}\left(\frac{\varepsilon}{a}\right)^{2l} \tag{C.2}$$

$$\forall n\in[0,\infty) \text{ and } m\in[-n,+n] \tag{C.3}$$

By expanding the factorials in (C.2), one can reach to following recursive formula, which is numerically stable especially for high degrees and orders n/m.

$$\lambda_{n,m,k} = \frac{((2n-4l+1)(n-2l-m+1)(n-2l-m+2)(n-2l+m+1)(n-2l+m+2))^{1/2}}{2k(2n-2l+1)(2n-4l+5)^{1/2}} \left(\frac{\varepsilon}{a}\right)^{2l} \lambda_{n,m,l-1}$$
$$\forall\, l \in [1,(n-m)/2],\ n \in [0,\infty) \text{ and } m \in [-n,+n] \tag{C.4}$$

with the start value

$$\lambda_{n,m,0} = 1 \quad \forall\, n, m \tag{C.5}$$